PLDP-IDS: Personalized Local Differential Privacy for Infinite Data Stream

Leilei Du [1], Yikun Hu [1], Xu Zhou [1], Yang Cao [2], Peng Cheng [3], Wangze Ni [4], Kenli Li [1]
[1]*Hunan University, Changsha, China;* [2]*Institute of Science Tokyo, Tokyo, Japan;*
[3]*Tongji University, Shanghai, China;* [4]*Zhejiang University, China*
leileidu@hnu.edu.cn; yikunhu@hnu.edu.cn; zhxu@hnu.edu.cn; cao@c.titech.ac.jp;
cspcheng@tongji.edu.cn; niwangze@zju.edu.cn; lkl@hnu.edu.cn

***Abstract*—Streaming data collection under local differential privacy (LDP) enables real-time analytics without a trusted curator. Existing w-event LDP mechanisms, however, impose a uniform privacy window and budget on all users, forcing everyone to adopt the most stringent requirement. We study personalized continual frequency estimation under LDP, where each user specifies her own window size and privacy budget, and identify two challenges. First, a single sampling size cannot balance sampling and perturbation errors for privacy groups with heterogeneous requirements. Second, borrowing estimates from groups with larger budgets reduces variance but introduces bias when privacy preferences correlate with the data. We first propose Optimal Population Selection (OPS) and two OPS-guided mechanisms, PLPD and PLPA. We then develop PLPD$^+$ and PLPA$^+$, which combine group-aware population scheduling with bias-aware cross-group fusion while preserving personalized w-event LDP. Experiments on real-world and synthetic streams show that PLPA$^+$ reduces the AMSE of the direct personalized extension of LDP-IDS by 52.3% and 40.8% on average, respectively, and remains about $5\times$ more accurate than PLPA under strong privacy–data correlation.**

***Index Terms*—local differential privacy, data stream, personalized differential privacy.**

I. Introduction

Mobile devices, wearable sensors, and location-based services continuously generate user data streams that support real-time analytics such as traffic monitoring, event detection, and mobility-aware recommendation [1], [2]. A fundamental task behind these applications is continual frequency estimation, where a collector estimates the population-level distribution of user values (e.g., the regions where drivers are located) at every time slot. Collecting raw streams, however, can reveal sensitive individual patterns: consecutive location reports may expose daily routines or repeated visits to hospitals, religious facilities, or private residences. Local Differential Privacy (LDP) [3], [4] addresses this risk without a trusted curator by letting each user perturb her data before reporting.

Unlike static data, consecutive reports from the same user are temporally correlated, thus, protecting each event independently fails to hide patterns spanning multiple slots, whereas protecting the entire stream exhausts the privacy budget on infinite streams. w-event privacy [5] strikes a balance by protecting any event sequence within a sliding window of length w, and LDP-IDS [6] extends it to the local setting. Existing w-event LDP mechanisms, however, impose a uniform window size and privacy budget on all users. In practice, users differ in how long their behaviors remain correlated and in how sensitive their data are, and thus naturally demand different (w_i, ϵ_i). A uniform mechanism has to adopt the most stringent requirement for everyone, as the following car-hailing example shows.

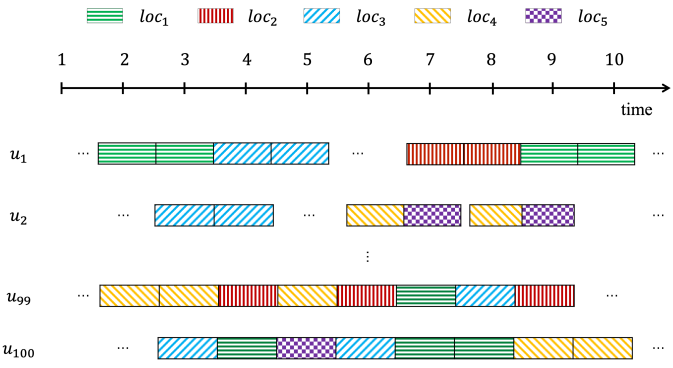


Fig. 1. An example for personalized w-event

Example 1. *As illustrated in Figure 1, consider* $n = 100$ *drivers over* $d = 5$ *regions, where* 98 *drivers require protection within* $w = 4$ *window length and the remaining two (*u_{99} *and* u_{100}*) require* $w = 8$*. A conventional uniform* w*-event LDP mechanism has to set* $w = 8$ *for all users. With privacy budget* $\epsilon = 1$*, its standard error is SE=*$\sqrt{Var_{FO}\left[\frac{\epsilon}{w}, n\right]/n^2}=\sqrt{\frac{d-2+e^{\epsilon/w}}{n(e^{\epsilon/w}-1)^2}}\approx 1.527$*, where* $Var_{FO}[\epsilon, m] = m \cdot \frac{d-2+e^{\epsilon}}{(e^{\epsilon}-1)^2}$ *[3]. However, if users are divided according to their personalized windows, using* $w_1 = 4$ *for the first group and* $w_2 = 8$ *for the second group, the standard error decreases to approximately SE=*$\sqrt{\left(Var_{FO}\left[\frac{\epsilon}{w_1}, n_1\right]+Var_{FO}\left[\frac{\epsilon}{w_2}, n_2\right]\right)/n^2}\approx 0.753$*. This example shows that a worst-case uniform window can cause substantial utility loss, motivating personalized continual LDP.*

In this paper, we study personalized continual frequency estimation under w-event personalized LDP, formalized as $(\boldsymbol{w}, \boldsymbol{\epsilon})$-EPLDP, where each user specifies her own window size w_i and privacy budget ϵ_i, and the collector estimates the population-level frequency distribution at every time slot. Personalized temporal privacy has been studied under central-

ized DP [7], however, those mechanisms inject noise into raw streams held by a trusted curator and thus do not apply to the local model.

Existing methods and limitations. A straightforward solution splits each user's budget across the reports in her window, which quickly shrinks the per-report budget under large w and infinite streams. LDP-IDS [6] avoids such fragmentation via population division: at each slot, only a sampled population reports with a larger budget, while the others are recycled or approximated by the latest release. However, LDP-IDS assumes a uniform (w, ϵ). Applying it with the most stringent requirement incurs the utility loss shown in Example 1.

Challenges. Directly extending LDP-IDS to heterogeneous requirements leaves two challenges unresolved:

Challenge 1: heterogeneous population scheduling. Population division couples sampling with privacy. A larger sampling size reduces the sampling error, however, shortens the recycling period, forcing users with long windows to split their budgets over more reports and thus increasing their perturbation error; a smaller size does the opposite. Since this trade-off differs across users with heterogeneous (w_i, ϵ_i) and across groups of different sizes, a single sampling size can only strike one compromise for all of them, over-sampling some groups and under-sampling others.

Challenge 2: biased cross-group fusion. Groups with small budgets suffer high estimation variance. Borrowing estimates from groups with larger budgets can reduce this variance, however, privacy preferences may correlate with the data: users with more sensitive data tend to choose stronger protection, thus, groups follow different distributions and naive borrowing introduces bias. When groups update asynchronously, the borrowed estimates may also be stale. Since the true group distributions are unknown, deciding how much to borrow is itself an estimation problem.

Our approaches. To address Challenge 1, we first derive the reporting error of population division under personalized budgets as the sum of sampling and perturbation errors, and propose Optimal Population Selection (OPS), which selects the privacy-compatible sampling size minimizing this error. Built on OPS, we propose two personalized population-division mechanisms, Personalized Local Population Distribution (PLPD) and Personalized Local Population Absorption (PLPA). Since a unified size still ignores group heterogeneity, we further propose Group-Aware Optimal Population Selection (GA-OPS), which assigns each privacy group its own sampling size, recycling period, and effective budget, so that each group follows its own error trade-off while satisfying its privacy requirement by construction.

To address Challenge 2, we propose Bias-Aware Personalized Utility Enhancement (BA-PUE). For each group, BA-PUE combines its own estimate with those borrowed from groups with weaker privacy requirements, and chooses the combination weights to minimize the resulting error, which accounts for both the variance of each estimate and its bias due to distribution shift or staleness. Since this error depends on the unknown group distributions, BA-PUE constructs an unbiased empirical risk matrix, and clips eigenvalues for fixed candidate inputs. After that, it uses a variance-corrected plug-in risk to select data-dependent fusion weights. The fused group estimates are then aggregated by population proportions. As a result, BA-PUE borrows heavily when groups are similar and lightly when the shift grows. Integrating GA-OPS and BA-PUE yields the enhanced mechanisms PLPD^+ and PLPA^+.

Contributions. In summary, we make the following contributions.

- We formalize the problem of Personalized Private Streaming Data Estimation in Local Setting under a newly defined $(\boldsymbol{w}, \boldsymbol{\epsilon})$-EPLDP, where each user specifies her own window size and privacy budget in Section III.
- We analyze the error of population-division under personalized LDP and propose PLPD and PLPA, guided by a new Optimal Population Selection (OPS) strategy that selects a privacy-compatible sampling size, together with their privacy and error analysis in Section IV.
- We propose GA-OPS for group-specific scheduling and BA-PUE for bias-aware cross-group fusion, yielding PLPD^+ and PLPA^+, and establish their privacy guarantees and error bounds in Section V.
- We conduct experiments on two real and three synthetic datasets, which shows that PLPA^+ reduces the AMSE of Direct-PLPA by 52.3% and 40.8% on average, respectively in Section VI. Besides, a controlled study further shows that PLPA^+ with BA-PUE remains about $5\times$ more accurate than PLPA under strong privacy-data correlation.

II. Related Work

A. Local Differential Privacy on Streaming Data

Local Differential Privacy (LDP) [4] enables privacy-preserving data collection without a trusted curator and has been widely adopted in practice. Existing studies on LDP for streaming data can be broadly categorized into event-level, user-level, and w-event privacy models.

Event-level LDP (event-LDP) mechanisms, such as RAPPOR [8] and ToPL [1], perturb each record independently and do not account for temporal correlations. User-level LDP methods [2], [9]–[11] aim to protect entire user streams, but typically suffer from rapid privacy budget consumption or limited adaptability in infinite streams. To bridge the gap between event-level and user-level protection, w-event privacy was introduced to protect event sequences within a sliding window of length w [5]. Ren et al. [6] extend this notion to the local setting and propose LDP-IDS for infinite data streams. However, LDP-IDS assumes a uniform window size and privacy budget for all users, which limits its applicability in personalized settings. SHieLD combines difference-tree reporting, sampling-based replication, and cross-level weighting for personalized continual LDP [12]. It does not model user-specific w_i-event windows or privacy–data correlation, and is therefore complementary rather than directly comparable.

TABLE I
SUMMARY FOR RELATED WORK.

Categories	Model Types	Methods	Infinite & correlated	Trust-free	Personalized
LDP on streaming data	event-level LDP	RAPPOR [8]	×	✓	×
		ToPL [1]	×	✓	×
	user-level LDP	CGM [9]	✓	✓	×
		DDRM [2]	✓	✓	×
		StaSwitch [10]	✓	✓	×
		PrivShape [11]	✓	✓	×
	w-event LDP	LDP-IDS [6]	✓	✓	×
Personalized DP	item-grained privacy	OSDP [14]	×	×	✓
		Pufferfish [15], [24]–[27]	✓	×	✓
		Blowfish [16], [28], [29]	✓	×	✓
		PRR [30]	×	✓	✓
	user-grained privacy	SM [17]	×	×	✓
		AdaPDP [18]	×	×	✓
		ADPM [19]	×	×	✓
		PCE [20]	×	✓	✓
		HDG-COE [22]	×	✓	✓
		PLDP-MRQ [21]	×	✓	✓
Our methods			✓	✓	✓

B. Personalized Differential Privacy

Personalized Differential Privacy (PDP), also known as Heterogeneous Differential Privacy (HDP) [13], allows heterogeneous privacy requirements across users or data attributes. Existing PDP models can be broadly divided into item-grained and user-grained settings.

Item-grained Privacy. Item-grained approaches, such as One-Sided Differential Privacy (OSDP) [14], Pufferfish [15] and Blowfish [16], relax protection for less sensitive attributes to improve utility. However, they rely on centralized mechanisms and policy specification, making them unsuitable for decentralized streaming settings.

User-grained Privacy. User-grained PDP allows each user to specify an individual privacy budget. Representative works include Personalized Differential Privacy (PDP) [17], Adaptive Personalized Differential Privacy (AdaPDP) [18], and Affine Differentially Private Mean Estimator (ADPM) [19], which primarily focus on centralized or static data. More recently, Chen et al. [20] and He et al. [21] propose Personalized Local Differential Privacy (PLDP) frameworks for count and range queries. Li et al. [22] propose HDG-COE, which enables users to specify personalized indistinguishable areas for protection, achieving high utility. However, existing PLDP methods do not consider temporal correlations in infinite data streams.

Beyond LDP and PLDP, several studies investigate streaming data privacy under centralized DP with a trusted curator, including w-event privacy [5], ℓ-trajectory privacy [23], and $(\boldsymbol{w}, \boldsymbol{\epsilon})$-personalized privacy [7]. In contrast, we study personalized local differential privacy for infinite, correlated attribute streams without requiring any trusted curator. Table I summarizes representative studies.

III. PROBLEM SETTINGS

In this section, we first introduce key conceptions related to data stream. Next, we present our new privacy definition: $\boldsymbol{w}$-Event $\boldsymbol{\epsilon}$-Personalized Local Differential Privacy $\big((\boldsymbol{w}, \boldsymbol{\epsilon})$-EPLDP$\big)$ based on the conception of Personalized Local Differential Privacy (PDP). Finally, we provide the problem definition: Personalized Private Streaming Data Estimation in Local Setting (PPSDELS). Table II summarizes the common notions used in this paper.

TABLE II
NOTATIONS.

Notations	Description
u_i; n	the i-th user; the total number of users
d	the domain size of the attribute
w_i; ϵ_i	u_i's privacy window size; u_i's privacy budget
PG_r	the r-th privacy group $(\widetilde{w}_r, \widetilde{\epsilon}_r, U_r)$
m, K	the total number of privacy groups grouped by privacy budgets or privacy requirements
π_r	the population proportion of privacy group PG_r
ω_j	the j-th value in the value domain
$\boldsymbol{f}$	the true population-level frequency histogram at any time slot
$\boldsymbol{f}_t$	the true population-level frequency histogram at time slot t
$\widehat{\boldsymbol{f}}_t$	the estimated statistical population-level frequency histogram at time slot t
$\boldsymbol{f}_t^{(r)}$	the true frequency histogram of PG_r at time slot t
$\widehat{\boldsymbol{f}}_t^{(r)}$	the estimated frequency histogram of PG_r at time slot t

A. Data Streams

We follow the standard streaming data model [5], [31]. At each time slot t, users report attribute values from a finite domain Ω of size d, forming a database D_t. The sequence $S = (D_1, D_2, \dots)$ constitutes an infinite data stream. The server aims to estimate the frequency distribution $\boldsymbol{f}_t$ at each time slot under local privacy constraints.

Definition 1 (w-Neighboring Stream Prefixes [5], [31])**.** Let w be a positive integer. Two stream prefixes S_t, S'_t are w-neighboring (i.e., $S_t \sim_w S'_t$), if:

1) for each $S_t[\tau]$, $S'_t[\tau]$ such that $\tau \le t$ and $S_t[\tau] \neq S'_t[\tau]$, it holds that $S_t[\tau]$ and $S'_t[\tau]$ are neighboring [5] in centralized DP, and
2) for each $S_t[\tau_1]$, $S_t[\tau_2]$, $S'_t[\tau_1]$, $S'_t[\tau_2]$ with $\tau_1 \le \tau_2$, $S_t[\tau_1] \neq S'_t[\tau_1]$ and $S_t[\tau_2] \neq S'_t[\tau_2]$, it holds that $\tau_2 - \tau_1 + 1 \le w$.

B. Privacy Definitions

Let $U = \{u_1, \dots, u_n\}$ be the user set. Each user u_i is assigned a personalized privacy requirement (w_i, ϵ_i) [7], where w_i is the protected temporal-window length and ϵ_i is the privacy budget for that window.

Definition 2 ($\boldsymbol{w}$-Event $\boldsymbol{\epsilon}$-Personalized Local Differential Privacy, $(\boldsymbol{w}, \boldsymbol{\epsilon})$-EPLDP)**.** Let $\boldsymbol{\mathcal{R}} = \{\mathcal{R}_i \mid i \in [|U|]\}$ be a local randomizer set with which each user $u_i \in U$ perturbs her own stream $x_i = (x_{i,1}, x_{i,2}, \dots)$. $\boldsymbol{\mathcal{R}}$ satisfies $(\boldsymbol{w}, \boldsymbol{\epsilon})$-EPLDP if $\forall u_i \in U$, $\forall S_{i,t}, S'_{i,t}$ satisfying $S_{i,t} \sim_{w_i} S'_{i,t}$, and $\forall O \in \mathcal{O}$ of output sequences,

$$\Pr[\mathcal{R}_i(S_{i,t}) \in O] \le e^{\epsilon_i} \Pr[\mathcal{R}_i(S'_{i,t}) \in O],$$

where the probability is over $\mathcal{R}_i$ and the server's sampling.

When $w_i = 1$ for all users, the definition above reduces to $\boldsymbol{\epsilon}$-Personalized Local Differential Privacy ($\boldsymbol{\epsilon}$-PLDP) [21]. Additionally, when (w_i, ϵ_i) is a constant pair $\big($i.e., $(w, \epsilon)\big)$, it collapses to w-Event ϵ-LDP [5], [6]. Following References [17], [21], we treat (w_i, ϵ_i) as public metadata needed for scheduling: neighboring streams in Definition 2 differ only in data values, and protecting the requirement itself is orthogonal to this work.

In practice, personalized privacy requirements can be discretized into a finite set of window-budget pairs, allowing users with identical requirements to be assigned to the same privacy group.

Definition 3 (Privacy Group)**.** Given a unique privacy requirement set $(\widetilde{\boldsymbol{w}}, \widetilde{\boldsymbol{\epsilon}}) = \{(\widetilde{w}_j, \widetilde{\epsilon}_j) \mid \forall r \neq j, \widetilde{w}_r \neq \widetilde{w}_j \text{ or } \widetilde{\epsilon}_r \neq \widetilde{\epsilon}_j\}$ with $|(\widetilde{\boldsymbol{w}}, \widetilde{\boldsymbol{\epsilon}})| = K$ and a user set $U \subseteq \mathcal{U}$ with n users, where each user u_i holds a privacy requirement $(w_i, \epsilon_i) \in (\widetilde{\boldsymbol{w}}, \widetilde{\boldsymbol{\epsilon}})$, U can be divided into K groups $\{U_1, U_2, \dots U_K\}$ based on distinct privacy requirement. The pair $PG_j = (\widetilde{w}_j, \widetilde{\epsilon}_j, U_j)$ is called the j-th Privacy Group of U.

Based on Definition 3, we can define privacy domination in Definition 4, which means PG_j requires no-weaker privacy requirement than PG_i if PG_j privacy-dominates PG_i.

Definition 4 (Privacy Domination)**.** For any two Privacy Group $PG_i = (\widetilde{w}_i, \widetilde{\epsilon}_i, U_i)$ and $PG_j = (\widetilde{w}_j, \widetilde{\epsilon}_j, U_j)$, PG_j privacy-dominates PG_i denoted by $PG_i \preceq_p PG_j$, if $\widetilde{w}_i \leq \widetilde{w}_j$ and $\widetilde{\epsilon}_i \geq \widetilde{\epsilon}_j$.

Let $\boldsymbol{PG}$ be the total privacy groups. For any privacy group $PG_i \in \boldsymbol{PG}$, we define its *source* set as the set of privacy groups dominated by PG_i, denoted as $\mathcal{S}(i) = \{PG_r \mid PG_r \preceq_p PG_i, PG_r \neq PG_i, PG_r \in \boldsymbol{PG}\}$, which is abbreviated as $\mathcal{S}_i$.

C. Problem Definition

Based on the above definitions, the server observes only locally perturbed reports and reconstructs an estimated frequency stream $(\widehat{\boldsymbol{f}}_1, \widehat{\boldsymbol{f}}_2, \dots)$. The goal is to minimize estimation error while satisfying $(\boldsymbol{w}, \boldsymbol{\epsilon})$-EPLDP.

Definition 5. (PPSDELS Problem). Given any positive integer $T \in \mathbb{N}^+$, a user set $U = \{u_1, u_2, ..., u_n\} \in \mathcal{U}$, where each u_i holds a privacy requirement pair (w_i, ϵ_i) and a series of data $x_{i,t}$ for $t \in [T]$, all the $x_{i,t}$ at time slot t form D_t, and all the D_t form a data stream prefix $S_T = (D_1, D_2, \dots, D_T)$. PPSDELS is to publish an obfuscated stream prefix frequency estimation $(\widehat{\boldsymbol{f}}_1, \widehat{\boldsymbol{f}}_2, \dots, \widehat{\boldsymbol{f}}_T)$ of S_T achieving $(\boldsymbol{w}, \boldsymbol{\epsilon})$-EPLDP with the error between $\widehat{\boldsymbol{f}}$ and $\boldsymbol{f}$ minimized, namely:

$$\min \sum_{t \in [T]} \mathbb{E}\left[\left\|\widehat{\boldsymbol{f}}_t - \boldsymbol{f}_t\right\|_2^2\right]$$
$$\text{s.t.} \sum_{\tau = \max(t - w_i + 1, 1)}^{t} \epsilon_{i,\tau} \leq \epsilon_i, \quad \forall u_i \in U,$$

where $\epsilon_{i,\tau}$ denotes the privacy budget consumption of u_i at time slot τ.

IV. OPS-Guided Personalized Population Division

In this section, we study the case where privacy requirements are independent of user values, thus, all privacy groups share the same population distribution. The case privacy requirements and user values are correlated is considered in Section V.

Following the population-division paradigm [3], [6], [32] , we first establish the frequency-estimation error under personalized LDP and use it to derive a unified Optimal Population Selection (OPS) strategy. Based on OPS, we then instantiate two basic personalized population-division mechanisms, Personalized Local Population Distribution (PLPD) and Personalized Local Population Absorption (PLPA).

Algorithm 1: Optimal Population Selection (OPS)

Input: The sampling size requirement list $\boldsymbol{z} = (z_1, \dots z_n)$; the privacy budget requirement list $\boldsymbol{\epsilon} = (\epsilon_1, \dots \epsilon_n)$; the window size list $\boldsymbol{w} = (w_1, \dots, w_n)$; the sampling ratio ρ; the value domain size d

Output: The optimal sampling size z^*; the optimal recycle period τ^*; the effective privacy budget vector $\boldsymbol{\epsilon}'$

```
Initialize err* as the upper bound of error value;
Set z̃ ← Unique(z) as the unique list of z;
for z̃ ∈ z̃ do
    Set τ̃ ← max(1, ⌊ρn/z̃⌋);
    Initialize ε'' ← ∅;
    for i ∈ [n] do
        ε_eff,i ← ε_i / ⌈w_i/τ̃⌉;
        Add ε_eff,i to ε'';
    Set ε̃'' ← Unique(ε'');
    Calculate α ← |{i : ε_eff,i = ε̃_r}|/n;
    Get the sampling size set z' = {z'_r | z'_r = z̃ · α_r};
    Calculate err_tmp ← err(ε̃'', n, z', d) according to
      Equation (1);
    if err_tmp < err* then
        err* ← err_tmp; z* ← z̃; τ* ← τ̃; ε' ← ε'';
return z*, τ*, ε';
```

A. Personalized Frequency Oracle Estimation and Error

We use a GRR-based Personalized Frequency Oracle (PFO) to combine the unbiased estimate $\widehat{f}_j^{(r)}$ from each budget group r with z_r sampling users' reports using inverse average-variance weighting [21]:

$$\forall \widetilde{f}_j \in \widetilde{\boldsymbol{f}}, \widetilde{f}_j = \sum_{r=1}^{m} \eta_r \cdot \widehat{f}_j^{(r)}, \quad \eta_r = \frac{1/\,\overline{\text{Var}}\left[\widehat{f}_j^{(r)}; \boldsymbol{z}\right]}{\sum_{r'=1}^{m} 1/\,\overline{\text{Var}}\left[\widehat{f}_j^{(r')}; \boldsymbol{z}\right]},$$

where $p_r = e^{\epsilon_r}/(e^{\epsilon_r} + d - 1)$ and $q_r = 1/(e^{\epsilon_r} + d - 1)$ are the truthful and fixed nontruthful report probabilities, respectively, and $\overline{\text{Var}}[\widehat{f}_j^{(r)}; \boldsymbol{z}] = \frac{q_r(1-q_r)}{z_r(p_r-q_r)^2} + \frac{1-p_r-q_r}{z_r(p_r-q_r)} \cdot \frac{1}{d}$ is an average approximating value of LDP unbiased variance [3].

Let $z_{sum} = \sum_{r \in [m]} z_r$ denote the total sampling size. The mechanism error consists of two components: (1) the PFO *perturbation variance* $\mathrm{V}_{PFO}(\widetilde{\boldsymbol{\epsilon}}, \boldsymbol{z}, d)$ introduced by personalized local randomization; and (2) the *sampling variance* $\text{Var}_{smpl}[\widehat{\boldsymbol{f}}]$ caused by using only a subset of the population. For the PFO perturbation variance, we have $\mathrm{V}_{PFO}(\widetilde{\boldsymbol{\epsilon}}, \boldsymbol{z}, d) = 1/\sum_{r=1}^{m} z_r/(d\lambda_r + \mu_r)$, where $\lambda_r = \frac{q_r(1-q_r)}{(p_r-q_r)^2}$ and $\mu_r = \frac{1-p_r-q_r}{(p_r-q_r)}$. For the sampling variance, we have $\text{Var}_{smpl}[\widehat{\boldsymbol{f}}] = \frac{n-z_{sum}}{z_{sum}(n-1)}(1 - \sum_{j=1}^{d} f_j^2) \leq \frac{n-z_{sum}}{z_{sum}(n-1)}$.

Let $\mathrm{V}_{smpl}(n, z_{sum}) = \frac{n-z_{sum}}{z_{sum}(n-1)}$. We use the following distribution-free upper bound on the per-domain reporting error:

$$\begin{aligned} err(\widetilde{\boldsymbol{\epsilon}}, n, \boldsymbol{z}, d) &= \left(\mathrm{V}_{smpl}(n, z_{sum}) + \mathrm{V}_{PFO}(\widetilde{\boldsymbol{\epsilon}}, \boldsymbol{z}, d)\right)/d \\ &= \frac{n - z_{sum}}{d z_{sum}(n-1)} + 1/\sum_{r=1}^{m} z_r/(\lambda_r + \mu_r/d), \end{aligned} \tag{1}$$

Detailed derivations of the PFO perturbation and sampling variances are provided in Appendix IX-A1.

B. Unified OPS: Dissimilarity and Population Selection

The core idea of population division is to determine whether the current distribution should be newly estimated or represented by the latest publication. Specifically, a new estimate is generated when the private dissimilarity *dis* exceeds the reporting error *err* defined in Equation (1); otherwise, the latest publication is reused.

Dissimilarity. Following Reference [6], we use the variance-corrected private dissimilarity

$$dis = \frac{1}{d}\sum_{j=1}^{d}\left(\widehat{\boldsymbol{f}}_{s,t}[j] - \widehat{\boldsymbol{f}}_{l}[j]\right)^2 - \frac{1}{d}\sum_{j=1}^{d} Var\left[\widehat{\boldsymbol{f}}_{s,t}[j]\right].$$

This correction removes the estimation variance of the current private estimate; its detailed derivation is provided in Appendix IX-B.

However, the sampling size must also satisfy users' personalized w-event privacy constraints, while an overly conservative choice may incur unnecessary sampling error.

To address this issue, we introduce Optimal Population Selection (OPS), which evaluates privacy-compatible candidate sampling sizes and selects the one minimizing the reporting error in Equation (1):

$$z_{opt} = \arg\min_{\boldsymbol{z}\in\mathcal{Z}} err(\widetilde{\boldsymbol{\epsilon}}', n, \widetilde{\boldsymbol{z}}', d),$$

where $\mathcal{Z}$ is the candidate sampling-size set, and $\widetilde{\boldsymbol{\epsilon}}'$ and $\widetilde{\boldsymbol{z}}'$ are the effective privacy budgets and corresponding group-wise sampled populations induced by $\boldsymbol{z}$. This process is illustrated in Algorithm 1.

For each candidate $\widetilde{z}$, OPS sets the recycling period as $\widetilde{\tau} = \max\{1, \lfloor \rho n/\widetilde{z} \rfloor\}$ (Line 4), so that user u_i uses the effective privacy budget $\epsilon_{eff,i} = \epsilon_i/\lceil w_i/\widetilde{\tau}\rceil$ (Line 7). OPS then evaluates Equation (1) under the resulting effective-budget groups and records the candidate with minimum error (Line 14).

C. OPS-Guided PLPD and PLPA

At initialization, the available user set U_A is set as the total users U. Both PLPD and PLPA follow Algorithm 2. They share the same OPS, dissimilarity, and publication decision. They differ only in the publication schedule: for PLPD, the potential publication population is $n_{pp}^{\mathrm{PLPD}} = \lfloor (n/2 - \sum_{\tau=t-\tau^*+1}^{t-1} |U_{pub,\tau}|)/2 \rfloor$; for PLPA, let $t_N = |U_{pub,l}|/z^* - 1$. If $t - l \le t_N$, the latest publication is reused. Otherwise, $n_{pp}^{\mathrm{PLPA}} = z^* \min\{t - l - t_N, \tau^*\}$. For a candidate group-size vector $\boldsymbol{z}_{pp}$, its potential publication error is $err_{pub} = err(\widetilde{\boldsymbol{\epsilon}}_{pub}, n, \boldsymbol{z}_{pp}, d)$, where $\boldsymbol{z}_{pp}$ is obtained from the privacy-group composition of the candidate population. Algorithm 2 summarizes the common workflow.

D. Analysis

Time Complexity Analysis. Let $m_{eff} = \left|\mathrm{Unique}\left(\boldsymbol{\epsilon}'_{opt}\right)\right| \le n$ denote the number of effective privacy budget groups after OPS.

Theorem IV.1. *The time complexity of both* PLPD *and* PLPA *is* $O\left(n + d \cdot m_{eff}\right)$.

Algorithm 2: Unified PLPD/PLPA Stream Estimation

Input: Available user set U_A; privacy requirements $(\boldsymbol{w}, \boldsymbol{\epsilon})$; history $(\widehat{\boldsymbol{f}}_1, \ldots, \widehat{\boldsymbol{f}}_{t-1})$; mode $\in \{\mathrm{PLPD}, \mathrm{PLPA}\}$

Output: Current estimate $\widehat{\boldsymbol{f}}_t$

1 $\boldsymbol{z} \leftarrow \left(\lfloor \frac{n}{2w_1} \rfloor, \lfloor \frac{n}{2w_2} \rfloor, \ldots, \lfloor \frac{n}{2w_n} \rfloor\right)$;
2 $(z^*, \tau^*, \boldsymbol{\epsilon}') \leftarrow \mathrm{OPS}(\boldsymbol{z}, \boldsymbol{\epsilon}, \boldsymbol{w}, 1/2, d)$ by Algorithm 1;
3 Sample the dissimilarity population $U_{dis,t}$ of size z^* and set $U_A \leftarrow U_A \backslash U_{dis,t}$;
4 $(\widehat{\boldsymbol{f}}_{dis,t}, \widehat{V}_{dis,t}) \leftarrow \mathrm{PFO}(D_{dis,t}; \boldsymbol{\epsilon}')$;
5 $dis \leftarrow \frac{1}{d}\|\widehat{\boldsymbol{f}}_{dis,t} - \widehat{\boldsymbol{f}}_{l}\|_2^2 - \frac{1}{d}\widehat{V}_{dis,t}$;
6 $(n_{pp}, status, err_{pub}) \leftarrow \mathrm{PubSchedule}(\mathsf{mode})$;
7 **if** *status* $=$ reuse **then**
8 $\quad \widehat{\boldsymbol{f}}_{pub,t} \leftarrow \widehat{\boldsymbol{f}}_{pub,t-1}$;
9 **else if** $dis > err_{pub}$ **and** $n_{pp} \ge u_{min}$ **then**
10 $\quad$ Sample $U_{pub,t}$ of size n_{pp} and set $U_A \leftarrow U_A \backslash U_{pub,t}$;
11 $\quad (\widehat{\boldsymbol{f}}_{pub,t}, \widehat{V}_{pub,t}) \leftarrow \mathrm{PFO}(D_{pub,t}; \boldsymbol{\epsilon}')$;
12 **else**
13 $\quad \widehat{\boldsymbol{f}}_{pub,t} \leftarrow \widehat{\boldsymbol{f}}_{pub,t-1}$;
14 Recycle expired users at time slot $t - \tau^* + 1$;
15 **return** $\widehat{\boldsymbol{f}}_{pub,t}$

Proof Sketch. OPS is executed once at initialization; at each slot, sampling and perturbation take $O(n)$ time and PFO estimation takes $O(d\, m_{eff})$ time. Therefore, the total complexity is $O(n + d\, m_{eff})$. The details are shown in Appendix IX-D1. □

Privacy Analysis. Theorem IV.2 gives the privacy guarantee for PLPD and PLPA.

Theorem IV.2. PLPD *and* PLPA *satisfy* $(\boldsymbol{w}, \boldsymbol{\epsilon})$*-EPLDP.*

Proof. Under OPS, each user participates at most once in every τ_{opt} consecutive slots. Hence, a w_i-window contains at most $c_i = \lceil w_i/\tau_{opt} \rceil$ reports. Each report uses $\epsilon'_i = \epsilon_i/c_i$, so sequential composition gives a total privacy loss no larger than ϵ_i. Dissimilarity computation, reuse, absorption, aggregation, and population selection are post-processing, which do not consume privacy budgets. Thus PLPD and PLPA satisfy $(\boldsymbol{w}, \boldsymbol{\epsilon})$-EPLDP. The details are shown in Appendix IX-E1. □

Utility Analysis. Let τ_{opt} denote the optimal recycling period determined by OPS. For analytical simplicity, we assume that at most $s_{opt} < \tau_{opt}$ new publications occur within a τ_{opt}-length window and that these publications are uniformly distributed. Let $\widetilde{\boldsymbol{\epsilon}}'_{opt} = \mathrm{Unique}(\boldsymbol{\epsilon}'_{opt})$ denote the distinct effective privacy budget obtained after OPS, and let $\epsilon_{opt,min} = \min(\widetilde{\boldsymbol{\epsilon}}'_{opt})$ denote the minimum effective privacy budget. The following bounds are on the expected 2-norm error.

Theorem IV.3. *The error upper bound of per time slot in* PLPD *is* $3R_{opt}\sqrt{\frac{nd}{z_{opt}}\left(\frac{1}{n-1} + A_{opt}\right) - \frac{d}{n-1}} + \frac{4(2^{s_{opt}}-1)}{s_{opt}}\left(\frac{1}{n-1} + A_{opt}\right) - \frac{1}{n-1}$, *where* $A_{opt} = \frac{d-1}{n} \cdot \frac{2e^{\epsilon_{opt,min}} + d - 2}{(e^{\epsilon_{opt,min}} - 1)^2}$ *and* $R_{opt} = \frac{e^{\epsilon_{opt,min}} + d - 2}{e^{\epsilon_{opt,min}} - 1}$.

Proof. Please refer to the details in Appendix IX-F1. □

Algorithm 3: Group-Aware Optimal Population Selection (GA-OPS)

Input: Privacy groups $\boldsymbol{PGT} = \{(\widetilde{w}_r, \widetilde{\epsilon}_r, n_r)\}_{r=1}^{K}$; sampling ratio ρ; domain size d
Output: Sampling-size vector $\boldsymbol{z}^*$; recycle-period vector $\boldsymbol{\tau}^*$; effective-budget vector $\boldsymbol{\epsilon}'$

for $r \in [K]$ **do**
  $N_r \leftarrow \lfloor \rho n_r \rfloor$;
  Construct $\mathcal{Z}_r \leftarrow \{\lfloor \omega N_r / \widetilde{w}_r \rfloor \mid \omega \in [\widetilde{w}_r]\} \cap [2, N_r]$;
  for $z_r \in \mathcal{Z}_r$ **do**
    $\tau_r \leftarrow \max\{1, \lfloor N_r / z_r \rfloor\}$; $c_r \leftarrow \lceil \widetilde{w}_r / \tau_r \rceil$;
    $\epsilon'_r \leftarrow \widetilde{\epsilon}_r / c_r$;
    Evaluate $err_r(z_r)$ and retain the minimizing candidate;
return $\boldsymbol{z}^*, \boldsymbol{\tau}^*, \boldsymbol{\epsilon}'$;

Theorem IV.4. *The error upper bound of per time slot in* PLPA *is* $3R_{opt}\sqrt{\frac{nd}{z_{opt}}(\frac{1}{n-1}+A_{opt})-\frac{d}{n-1}} + \frac{nC}{z_{opt}}(\frac{1}{n-1}+A_{opt})H_{B+1} - \frac{C(B+1)}{n-1} + BC \cdot err_{nlf}$, *where* $A_{opt}=\frac{d-1}{n}\frac{2e^{\epsilon_{opt,min}}+d-2}{(e^{\epsilon_{opt,min}}-1)^2}$, $R_{opt}=\frac{e^{\epsilon_{opt,min}}+d-2}{e^{\epsilon_{opt,min}}-1}$, $B=\frac{\tau_{opt}-s_{opt}}{2s_{opt}}$, $C=\frac{1}{2B+1}$, $H_{B+1}=\sum_{j=1}^{B+1}\frac{1}{j}$ *and* err_{nlf} *denotes the average approximation error of a nullified time slot.*

Proof. Please refer to the details in Appendix IX-F1. □

V. Enhanced Approaches

Although PLPD and PLPA provide personalized continual publication through OPS-guided population division, two limitations remain. First, unified population scheduling cannot fully accommodate group-specific privacy requirements and population sizes. Second, cross-group estimation may introduce composition bias when privacy groups follow heterogeneous distributions. To address these limitations, we develop GA-OPS for group-specific scheduling and BA-PUE for bias-aware cross-group estimation, yielding the enhanced mechanisms PLPD$^+$ and PLPA$^+$.

A. Group-Aware Optimal Population Selection (GA-OPS)

For each privacy group PG_r, GA-OPS reserves $N_r = \lfloor \rho n_r \rfloor \geq 2$ users and searches the candidate sampling sizes specified in Algorithm 3. Each candidate z_r induces a feasible recycling period and effective per-report privacy budget, and is evaluated by the average per-domain error $err_r(z_r) = \frac{1}{d}\left(\mathrm{V}_{smpl}(N_r, z_r) + \mathrm{V}_{LDP}(\epsilon'_r(z_r), z_r, d)\right)$. The minimum-error candidate determines $(z_r^*, \tau_r^*, \epsilon'_r)$ for group PG_r.

While GA-OPS improves population scheduling, it does not directly address estimation bias arising from heterogeneous group distributions. We therefore introduce BA-PUE to optimize cross-group information reuse.

B. Bias-Aware Personalized Utility Enhancement (BA-PUE)

Cross-group reuse can reduce group-level estimation variance [21], [33]. However it may introduce distribution-shift bias. Thus, we design BA-PUE, which uses Partial-Absorption-Aware Corrected Weight (PAA-CW) to construct target-specific simplex weights from group-level estimates and their variances by minimizing a corrected bias–variance risk. BA-PUE then applies these weights and aggregates the fused estimates according to population proportions.

Group-Level Inputs and Variance. For each privacy group PG_r, let N_r be the reserved population selected by GA-OPS, and let $\widehat{\boldsymbol{f}}^{(r)}$ be its unbiased GPRR estimate obtained from z_r reports under ϵ'_r ($2 \leq z_r \leq N_r$). Define $V_r \triangleq \sum_{j=1}^{d} \mathrm{Var}[\widehat{\boldsymbol{f}}^{(r)}[j]]$.

Theorem V.1. $\widehat{V}_r = \frac{\theta_r\left(1-\left\|\widehat{\boldsymbol{f}}^{(r)}\right\|_2^2\right)+V_{LDP,r}}{1-\theta_r}$ *is an unbiased estimator of* V_r, *i.e.,* $\mathbb{E}\left[\widehat{V}_r\right] = V_r$, *where* $\theta_r = \frac{N_r - z_r}{z_r(N_r-1)}$ *and* $V_{LDP,r} = V_{LDP}(\epsilon'_r, z_r, d)$.

Proof. (Sketch) Decomposing the error into finite-population sampling and local-randomization components yields $\mathbb{E}[\widehat{V}_r] = V_r$. The details are shown in Appendix IX-C1. □

Target-Specific Bias–Variance Objective. For each target group PG_H, BA-PUE combines its own candidate estimate with the estimates from its source set $\mathcal{S}_H$. Let $\mathcal{C}_H = \mathcal{S}_H \cup \{PG_H\}$. Because a candidate may represent a current or historical distribution, let $\boldsymbol{g}_t^{(r)}$ denote the distribution represented by the candidate of PG_r at time t. Each candidate provides an unbiased estimate $\widehat{\boldsymbol{g}}_{risk,t}^{(r)} = \boldsymbol{g}_t^{(r)} + \boldsymbol{e}_{risk,r,t}$, where $\mathbb{E}\left[\boldsymbol{e}_{risk,r,t}\right] = \boldsymbol{0}$ and $V_{risk,r,t} = \mathbb{E}\left[\|\boldsymbol{e}_{risk,r,t}\|_2^2\right]$.

For simplex weights $\boldsymbol{\lambda}_{H,t} \succeq \boldsymbol{0}$ with $\boldsymbol{1}^\top \boldsymbol{\lambda}_{H,t} = 1$, the enhanced estimate is

$$\widetilde{\boldsymbol{f}}_t^{(H)} = \sum\nolimits_{PG_r \in \mathcal{C}_H} \lambda_{H,r,t} \cdot \widehat{\boldsymbol{g}}_{risk,t}^{(r)}. \tag{2}$$

Let $\boldsymbol{\xi}_{r|H,t} = \boldsymbol{g}_t^{(r)} - \boldsymbol{f}_t^{(H)}$ denote the distribution shift from candidate PG_r to target PG_H. Assuming zero-mean and mutually uncorrelated candidate errors, the target-specific fusion risk $R_H(\boldsymbol{\lambda}_{H,t}) = \mathbb{E}[\|\widetilde{\boldsymbol{f}}_t^{(H)} - \boldsymbol{f}_t^{(H)}\|_2^2]$ is

$$\begin{aligned} R_H(\boldsymbol{\lambda}_{H,t}) &= \left\|\sum\nolimits_{PG_r \in \mathcal{C}_H} \lambda_{H,r,t}\boldsymbol{\xi}_{r|H,t}\right\|_2^2 + \sum\nolimits_{PG_r \in \mathcal{C}_H} \lambda_{H,r,t}^2 V_{risk,r,t} \\ &= \boldsymbol{\lambda}_{H,t}^\top \boldsymbol{M}_{H,t} \boldsymbol{\lambda}_{H,t}, \end{aligned}$$

where

$$\boldsymbol{M}_{H,t}(r,s) = \left\langle \boldsymbol{\xi}_{r|H,t}, \boldsymbol{\xi}_{s|H,t} \right\rangle + V_{risk,r,t}\delta_{r,s} \tag{3}$$

and $\delta_{r,s} = 1$ if $r = s$ and 0 otherwise.

Empirical PAA-CW Solver. The ideal matrix $\boldsymbol{M}_{H,t}$ is unobservable because the real group distributions are unknown. We therefore design PAA-CW by using auxiliary unbiased estimates $\widehat{\boldsymbol{f}}_{est,t}^{(r)} = \boldsymbol{f}_{est,t}^{(r)} + \boldsymbol{e}_{est,t}$ and $\widehat{\boldsymbol{f}}_{tar,t}^{(H)} = \boldsymbol{f}_{tar,t}^{(H)} + \boldsymbol{e}_{tar,t}$ with variance estimates $\widehat{V}_{est,r,t}$ and $\widehat{V}_{tar,H,t}$ to construct an empirical risk matrix $\widehat{\boldsymbol{M}}_{H,t}$ and obtain the fusion weights.

For $PG_r \in \mathcal{C}_H$, define $\widehat{\boldsymbol{\xi}}_{r|H,t} = \widehat{\boldsymbol{f}}_{est,t}^{(r)} - \widehat{\boldsymbol{f}}_{tar,t}^{(H)}$. Under mutually uncorrelated estimation errors between different groups, we have

$$\begin{aligned} &\mathbb{E}\left[\left\langle \widehat{\boldsymbol{\xi}}_{r|H,t}, \widehat{\boldsymbol{\xi}}_{s|H,t} \right\rangle\right] \\ &= \left\langle \boldsymbol{\xi}_{r|H,t}, \boldsymbol{\xi}_{s|H,t} \right\rangle + V_{tar,H,t} + V_{est,r,t}\delta_{r,s} - E_{r,H} - E_{s,H}, \end{aligned}$$

where $E_{r,s} = \mathbb{E}[\langle \boldsymbol{e}_{est,t}^{(r)}, \boldsymbol{e}_{tar,t}^{(s)} \rangle]$. For each target group PG_H, we distinguish two cases according to the relationship between

its auxiliary group estimates and target estimate: (1) the two estimates coincide; and (2) their estimation errors are uncorrelated.

For **case** (1), since $\widehat{\boldsymbol{f}}_{est,t}^{(H)} = \widehat{\boldsymbol{f}}_{tar,t}^{(H)}$ we have

$$\mathbb{E}\left[\left\langle \widehat{\boldsymbol{\xi}}_{r|H,t}, \widehat{\boldsymbol{\xi}}_{s|H,t} \right\rangle\right] = \begin{cases} \left\langle \boldsymbol{\xi}_{r|H,t}, \boldsymbol{\xi}_{s|H,t} \right\rangle + V_{est,H,t} + V_{est,r,t}\delta_{r,s}, & \text{if } r,s \neq H, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Then for $\boldsymbol{M}_{H,t}$ with $r \neq H$ and $s \neq H$, using unbiased estimator $\widehat{V}_{est,r,t}$, according to Equation (3) and (4), we have

$$\begin{aligned} \boldsymbol{M}_{H,t}(r,s) &= \left\langle \boldsymbol{\xi}_{r|H,t}, \boldsymbol{\xi}_{s|H,t} \right\rangle + V_{risk,r,t}\delta_{r,s} \\ &= \mathbb{E}\left[\left\langle \widehat{\boldsymbol{\xi}}_{r|H,t}, \widehat{\boldsymbol{\xi}}_{s|H,t} \right\rangle\right] - V_{est,H,t} - V_{est,r,t}\delta_{r,s} + V_{risk,r,t}\delta_{r,s} \\ &= \mathbb{E}\left[\left\langle \widehat{\boldsymbol{\xi}}_{r|H,t}, \widehat{\boldsymbol{\xi}}_{s|H,t} \right\rangle - \widehat{V}_{est,H,t} + \left(V_{risk,r,t} - \widehat{V}_{est,r,t}\right)\delta_{r,s}\right]. \end{aligned}$$

Thus, PAA-CW constructs

$$\widehat{\boldsymbol{M}}_{H,t}(r,s) = \begin{bmatrix} V_{risk,H,t} & \mathbf{0}^\top \\ \mathbf{0} & \widehat{\boldsymbol{A}}_{H,t} \end{bmatrix} \tag{5}$$

with

$$\widehat{A}_{H,t}(r,s) = \left\langle \widehat{\boldsymbol{\xi}}_{r|H,t}, \widehat{\boldsymbol{\xi}}_{s|H,t} \right\rangle - \widehat{V}_{est,H,t} + \left(V_{risk,r,t} - \widehat{V}_{est,r,t}\right)\delta_{r,s},$$

which satisfies $\mathbb{E}[\widehat{\boldsymbol{M}}_{H,t}] = \boldsymbol{M}_{H,t}$.

For **case** (2), we have $E_{r,H} = E_{s,H} = 0$, thus,

$$\mathbb{E}\left[\left\langle \widehat{\boldsymbol{\xi}}_{r|H,t}, \widehat{\boldsymbol{\xi}}_{s|H,t} \right\rangle\right] = \left\langle \boldsymbol{\xi}_{r|H,t}, \boldsymbol{\xi}_{s|H,t} \right\rangle + V_{tar,H,t} + V_{est,r,t}\delta_{r,s}. \tag{6}$$

Then for $\boldsymbol{M}_{H,t}$, using unbiased estimator $\widehat{V}_{tar,H,t}$ and $\widehat{V}_{est,r,t}$, according to Equation (3) and (6), we have

$$\begin{aligned} \boldsymbol{M}_{H,t}(r,s) &= \left\langle \boldsymbol{\xi}_{r|H,t}, \boldsymbol{\xi}_{s|H,t} \right\rangle + V_{risk,r,t}\delta_{r,s} \\ &= \mathbb{E}\left[\left\langle \widehat{\boldsymbol{\xi}}_{r|H,t}, \widehat{\boldsymbol{\xi}}_{s|H,t} \right\rangle\right] - V_{tar,H,t} - V_{est,r,t}\delta_{r,s} + V_{risk,r,t}\delta_{r,s} \\ &= \mathbb{E}\left[\left\langle \widehat{\boldsymbol{\xi}}_{r|H,t}, \widehat{\boldsymbol{\xi}}_{s|H,t} \right\rangle - \widehat{V}_{tar,H,t} + \left(V_{risk,r,t} - \widehat{V}_{est,r,t}\right)\delta_{r,s}\right]. \end{aligned}$$

Thus, PAA-CW constructs

$$\widehat{\boldsymbol{M}}_{H,t}(r,s) = \left\langle \widehat{\boldsymbol{\xi}}_{r|H,t}, \widehat{\boldsymbol{\xi}}_{s|H,t} \right\rangle - \widehat{V}_{tar,H,t} + \left(V_{risk,r,t} - \widehat{V}_{est,r,t}\right)\delta_{r,s} \tag{7}$$

which satisfies $\mathbb{E}[\widehat{\boldsymbol{M}}_{H,t}] = \boldsymbol{M}_{H,t}$.

Finite-sample fluctuations may make $\widehat{\boldsymbol{M}}_{H,t}$ indefinite. We therefore clip its eigenvalues at a fixed threshold $\mu > 0$: $\widetilde{\boldsymbol{M}}_{H,t} = \boldsymbol{Q}\,\mathrm{diag}(\max\{\nu_j, \mu\})\,\boldsymbol{Q}^\top \succeq \mu\boldsymbol{I}$. We denote this process as Positive Definite Eigenvalue Clipping (PDEC). PAA-CW invokes PDEC and then get the fusion weights by solving $\boldsymbol{\lambda}_{H,t}^* = \arg\min_{\boldsymbol{\lambda}\succeq\mathbf{0},\,\mathbf{1}^\top\boldsymbol{\lambda}=1} \boldsymbol{\lambda}^\top \widetilde{\boldsymbol{M}}_{H,t}\boldsymbol{\lambda}$. The process of PAA-CW is shown in Algorithm 4.

Group Fusion and Population-Level BA-PUE Output. Let $\pi_H = |\boldsymbol{U}_H|/n$ denote the population proportion of PG_H. Because privacy groups may have different distributions, BA-PUE aggregates the fused group estimates using π_H rather than inverse-variance weights. With $\gamma_{r,t} = \sum_{H=1}^K \pi_H \lambda_{H,r,t}^*$, the population estimate is

$$\widetilde{\boldsymbol{f}}_t = \sum_{H=1}^K \pi_H \cdot \widetilde{\boldsymbol{f}}_t^{(H)} = \sum_{r=1}^K \left(\sum_{H=1}^K \pi_H \lambda_{H,r,t}^*\right) \widehat{\boldsymbol{g}}_{risk,t}^{(r)} = \sum_{r=1}^K \gamma_{r,t} \cdot \widehat{\boldsymbol{g}}_{risk,t}^{(r)}.$$

For $\boldsymbol{f}_t = \sum_{r=1}^K \pi_r \boldsymbol{f}_t^{(r)}$, the average per-domain mean-squared error is

$$\begin{aligned} err_t^+ &= \frac{1}{d}\mathbb{E}\left[\left\|\widetilde{\boldsymbol{f}}_t - \boldsymbol{f}_t\right\|_2^2\right] \\ &= \frac{1}{d}\left(\sum_{r=1}^K \gamma_{r,t}^2 V_{risk,r,t} + \left\|\sum_{r=1}^K \gamma_{r,t} \cdot \boldsymbol{g}_t^{(r)} - \sum_{r=1}^K \pi_r \boldsymbol{f}_t^{(r)}\right\|_2^2\right). \end{aligned} \tag{8}$$

Algorithm 4: Partial-Absorption-Aware Corrected Weight (PAA-CW)

Input: $\widehat{\boldsymbol{F}}_{est}$; $\widehat{\boldsymbol{V}}_{est}$; $\widehat{\boldsymbol{F}}_{tar}$; $\widehat{\boldsymbol{V}}_{tar}$; $\boldsymbol{V}_{risk}$; total groups $\boldsymbol{PG}$; source sets $\boldsymbol{\mathcal{S}}$; clipping threshold μ; auxiliary status set $\boldsymbol{ST} = \{ST_r \mid PG_r \in \boldsymbol{PG}, ST_R \in \{\mathrm{EQ}, \mathrm{ND}\}\}$ for $\boldsymbol{PG}$

Output: Fusion weights $\boldsymbol{\Lambda}^*$

```
for PG_H ∈ PG do
    Set C_H ← {PG_H} ∪ S_H;
    if ST_H = EQ then
        Construct blocked M̂_H by Equation (5);
    else
        Construct general M̂_H by Equation (7);
    M̃_H ← PDEC(M̂_H, μ);
    Obtain λ*_H ← arg min_{λ⪰0, 1^⊤λ=1} λ^⊤ M̃_H λ;
    Add λ*_H to Λ*;
return Λ*
```

TABLE III
INPUTS SUPPLIED TO PAA-CW UNDER DIFFERENT PUBLICATION MODES.

Cases	$PG_r \in \boldsymbol{PG}_t^{(A)}$				
	$\widehat{\boldsymbol{f}}_{est}^{(r)}$	$\widehat{\boldsymbol{f}}_{tar}^{(r)}$	$\widehat{V}_{est,r}$	$\widehat{V}_{tar,r}$	$V_{risk,r}$
PLPD$^+$-Pre	$\widehat{\boldsymbol{f}}_{dis,t}^{(r)}$		$\widehat{V}_{dis,r,t}$		$V_{pub,r,t}^{\mathrm{Pre}}$
PLPD$^+$-Post	$\widehat{\boldsymbol{f}}_{pub,t}^{(r)}$		$\widehat{V}_{pub,r,t}$		$\widehat{V}_{pub,r,t}$
PLPA$^+$-Pre	$\widehat{\boldsymbol{f}}_{dis,t}^{(r)}$		$\widehat{V}_{dis,r,t}$		$V_{pub,r,t}^{\mathrm{Pre}}$
PLPA$^+$-Post	$\widehat{\boldsymbol{f}}_{pub,t}^{(r)}$		$\widehat{V}_{pub,r,t}$		$\widehat{V}_{pub,r,t}$
Cases	$PG_r \in \boldsymbol{PG}_t^{(R)}$				
	$\widehat{\boldsymbol{f}}_{est}^{(r)}$	$\widehat{\boldsymbol{f}}_{tar,r}$	$\widehat{V}_{est,r}$	$\widehat{V}_{tar,r}$	$V_{risk,r}$
PLPA$^+$-Pre	$\widehat{\boldsymbol{f}}_{pub,l_r}^{(r)}$	$\widehat{\boldsymbol{f}}_{dis,t}^{(r)}$	$\widehat{V}_{pub,r,l_r}$	$\widehat{V}_{dis,r,t}$	$\widehat{V}_{pub,r,l_r}$
PLPA$^+$-Post	$\widehat{\boldsymbol{f}}_{pub,l_r}^{(r)}$	$\widehat{\boldsymbol{f}}_{dis,t}^{(r)}$	$\widehat{V}_{pub,r,l_r}$	$\widehat{V}_{dis,r,t}$	$\widehat{V}_{pub,r,l_r}$

C. Personalized Local Population Enhanced Solutions

Both PLPD$^+$ and PLPA$^+$ use a variance-corrected plug-in estimate to decide whether a new population-level publication is necessary. The ideal risk err_t^+ in Equation (8) contains unknown current and historical group distributions. We first derive the observable risk for general mixed current–historical candidates used by PLPA$^+$ and then discuss the current candidates used by PLPD$^+$.

Risk Estimation. We divide the current privacy groups $\boldsymbol{PG}_t$ as two parts: (1) the *active groups* $\boldsymbol{PG}_t^{(A)}$; and (2) the *reused groups* $\boldsymbol{PG}_t^{(R)}$. Then err_t^+ can be further expressed as

$$\begin{aligned} err_t^+ = \frac{1}{d}\Bigg(&\sum_{r=1}^K \gamma_{r,t}^2 V_{risk,r,t} + \Bigg\| \sum_{PG_r \in \boldsymbol{PG}_t^{(A)}} (\gamma_{r,t} - \pi_r)\, \boldsymbol{f}_t^{(r)} \\ &+ \sum\nolimits_{PG_r \in \boldsymbol{PG}_t^{(R)}} \left(\gamma_{r,t}\boldsymbol{f}_{l_r}^{(r)} - \pi_r \boldsymbol{f}_t^{(r)}\right) \Bigg\|_2^2\Bigg). \end{aligned}$$

Let $B(\boldsymbol{f}_t, \boldsymbol{f}_{l_r}) = \sum_{PG_r \in \boldsymbol{PG}_t^{(A)}} (\gamma_{r,t} - \pi_r)\, \boldsymbol{f}_t^{(r)} + \sum_{PG_r \in \boldsymbol{PG}_t^{(R)}} (\gamma_{r,t}\boldsymbol{f}_{l_r}^{(r)} - \pi_r \boldsymbol{f}_t^{(r)})$. Let $\boldsymbol{H}_t = \{\boldsymbol{h}_t^{(r)} \mid r \in [K]\}$, where $\boldsymbol{h}_t^{(r)}$ is a proxy for $\boldsymbol{f}_t^{(r)}$, and let $\widehat{\boldsymbol{F}}_{l,t} = \{\widehat{\boldsymbol{f}}_{l_r}^{(r)} \mid PG_r \in$

$\boldsymbol{PG}_t^{(R)}\}$ contain the latest historical estimates of reused groups. For $B(\boldsymbol{f}_t, \boldsymbol{f}_{l_r})$, we have

$$\begin{aligned}&\|B\left(\boldsymbol{f}_t, \boldsymbol{f}_{l_r}\right)\|_2^2\\ =&\mathbb{E}\left[\left\|B\left(\boldsymbol{H}_t, \widehat{\boldsymbol{F}}_{l,t}\right)\right\|_2^2\right] - \sum\nolimits_{PG_r \in \boldsymbol{PG}_t^{(A)}} (\gamma_{r,t} - \pi_r)^2 \operatorname{Var}\left[\boldsymbol{h}_t^{(r)}\right]\\ &- \sum\nolimits_{PG_r \in \boldsymbol{PG}_t^{(R)}} \left(\pi_r^2 \operatorname{Var}\left[\boldsymbol{h}_t^{(r)}\right] + \gamma_{r,t}^2 \operatorname{Var}\left[\widehat{\boldsymbol{f}}_{l_r}^{(r)}\right]\right).\end{aligned}$$

By setting the variance-corrected squared-bias estimate as:

$$\begin{aligned}\widehat{B}_t^2 = \max\Big\{0, &\left\|B\left(\boldsymbol{H}_t, \widehat{\boldsymbol{F}}_{l,t}\right)\right\|_2^2 - \sum\nolimits_{r \in \boldsymbol{PG}_t^{(A)}} (\gamma_{r,t} - \pi_r)^2 \operatorname{Var}\left[\boldsymbol{h}_t^{(r)}\right]\\ &- \sum\nolimits_{r \in \boldsymbol{PG}_t^{(R)}} \left(\pi_r^2 \operatorname{Var}\left[\boldsymbol{h}_t^{(r)}\right] + \gamma_{r,t}^2 \operatorname{Var}\left[\widehat{\boldsymbol{f}}_{l_r}^{(r)}\right]\right)\Big\}.\end{aligned}$$

We can get the corrected error:

$$\widetilde{err}^{+}\left(\boldsymbol{H}_t, \boldsymbol{V}_{risk}\right) = \frac{1}{d}\left(\sum_{r \in \boldsymbol{PG}_t^{(A)}} \gamma_{r,t}^2 V_{risk,r}^{(A)} + \sum_{r \in \boldsymbol{PG}_t^{(R)}} \gamma_{r,t}^2 V_{risk,r}^{(R)} + \widehat{B}_t^2\right), \tag{9}$$

where $\boldsymbol{V}_{risk} = \boldsymbol{V}_{risk}^{(A)} \cup \boldsymbol{V}_{risk}^{(R)}$ is composed of the active variance set and reused variance set. Equation (9) gives a variance-corrected plug-in estimate for mixed current-and-historical publication candidates; it is not claimed to be strictly unbiased because the fusion weights and bias estimate depend on noisy data.

When $\boldsymbol{g}_t = \boldsymbol{f}_t$, $B\left(\boldsymbol{f}_t, \boldsymbol{f}_{l_r}\right)$ degrade into $B\left(\boldsymbol{f}_t\right) = \sum_{r=1}^{K} \left(\gamma_{r,t} - \pi_r\right) \boldsymbol{f}_t^{(r)}$. Consequently,

$$\widetilde{err}^{+}(\boldsymbol{H}_t, \boldsymbol{V}_{risk}) = \frac{1}{d}\left(\sum_{r=1}^{K} \gamma_{r,t}^2 V_{risk,r} + \widehat{B}_t^2\right), \tag{10}$$

where

$$\widehat{B}_t^2 = \max\left\{0, \left\|\sum_{r=1}^{K} (\gamma_{r,t} - \pi_r)\boldsymbol{h}^{(r)}\right\|_2^2 - \sum_{r=1}^{K} (\gamma_{r,t} - \pi_r)^2 \operatorname{Var}\left[\boldsymbol{h}^{(r)}\right]\right\}.$$

Both PLPD$^+$ and PLPA$^+$ combine GA-OPS and BA-PUE with personalized population division. They share the same dissimilarity-estimation stage and the same PAA-CW interface, however, differ in their publication schedules: PLPD$^+$ updates all privacy groups synchronously, whereas PLPA$^+$ updates active groups and reuses the latest publications for the remaining groups.

Personalized Local Population Distribution Plus: Synchronous Group Updates. In PLPD$^+$, the available user set U_A is initialized as total user set U. All groups provide current candidates, i.e., $\boldsymbol{g}_t^{(r)} = \boldsymbol{f}_t^{(r)}$, eliminating temporal-staleness bias. At each slot, GA-OPS determines group-specific reporting populations, and PAA-CW computes fusion weights using the inputs in Table III. A synchronous publication occurs only when *dis* $>$ $\widetilde{err}^{+}$ in Equation (10) and all groups satisfy the minimum-population requirement. Pre retains the planned fusion weights, whereas Post recomputes them from the realized publication reports. Algorithm 5 summarizes the procedure, where Π_Δ is the Euclidean projection onto the probability simplex.

Example 2. *Consider three privacy groups satisfying* $w_1 < w_2 < w_3$, $\epsilon_1 > \epsilon_2 > \epsilon_3$, *so that* $PG_1 \preceq_p PG_2 \preceq_p PG_3$, *with* $S_1 = \emptyset$, $S_2 = \{PG_1\}$, *and* $S_3 = \{PG_1, PG_2\}$. *Let* $d = 4$ *and* $\pi = (0.4, 0.35, 0.25)$. *GA-OPS independently determines* $(z_{opt,dis,r}^*, \tau_r^*, \epsilon_r')$ *for each group, yielding the group-specific population allocation illustrated in Figure 2. In* PLPD$^+$-Pre, *PAA-CW gives* $\lambda_1 = (1, 0, 0)$, $\lambda_2 = (0.35, 0.65, 0)$, *and* $\lambda_3 = (0.15, 0.30, 0.55)$, *yielding* $\gamma = (0.56, 0.303, 0.138)$. *With the illustrative estimates,* $dis = 0.0122 > \widetilde{err}^{+} = 0.00088$;

Algorithm 5: PLPD$^+$ Stream Estimation

Input: Available user set U_A; privacy requirements $(\boldsymbol{w}, \boldsymbol{\epsilon})$; history $(\widehat{\boldsymbol{f}}_1, \ldots, \widehat{\boldsymbol{f}}_{t-1})$; $\mathsf{mode}_{fsn} \in \{\mathrm{Pre}, \mathrm{Post}\}$
Output: $\widehat{\boldsymbol{f}}_t$

Initialize $\widehat{\boldsymbol{F}}_{pub,t}, \boldsymbol{V}_{pub,t}^{\mathrm{Pre}}, \widehat{\boldsymbol{V}}_{pub,t}, \widehat{\boldsymbol{F}}_{dis,t}, \widehat{\boldsymbol{V}}_{dis,t}$ as empty;
Construct $\boldsymbol{PG}$, $\boldsymbol{PGT}$, and $\boldsymbol{\mathcal{S}} \leftarrow \{\mathcal{S}_r \mid r \in [K]\}$;
// sub-mechanism $\mathcal{M}_{dis,t}$
$(\boldsymbol{z}_{opt,dis}^*, \boldsymbol{\tau}^*, \boldsymbol{\epsilon}') \leftarrow$ GA-OPS($\boldsymbol{PGT}, \frac{1}{2}, d$) (Algorithm 3);
for $PG_r \in \boldsymbol{PG}$ **do**
  Sample $U_{dis,r,t}$ of size $z_{opt,dis,r}^*$ from $U_{A,r}$;
  $U_{A,r} \leftarrow U_{A,r} \backslash U_{dis,r,t}$;
  $\widehat{\boldsymbol{f}}_{dis,t}^{(r)} \leftarrow$ FO($D_{dis,r,t}, \epsilon_r'$), and compute $\widehat{\boldsymbol{f}}_{dis,t}^{(r)}$'s unbiased vector-level variance estimate $\widehat{V}_{dis,r,t}$ by Theorem V.1;
  Add $\widehat{\boldsymbol{f}}_{dis,t}^{(r)}$ to $\widehat{\boldsymbol{F}}_{dis,t}$ and $\widehat{V}_{dis,r,t}$ to $\widehat{\boldsymbol{V}}_{dis,t}$;
Aggregate $\widehat{\boldsymbol{f}}_{dis,t} \leftarrow \sum_{r \in [K]} \pi_r \widehat{\boldsymbol{f}}_{dis,t}^{(r)}$;
$dis \leftarrow \frac{1}{d}\sum_{j=1}^{d}\left(\widehat{\boldsymbol{f}}_{dis,t}[j] - \widehat{\boldsymbol{f}}_{l}[j]\right)^2 - \frac{1}{d}\sum_{r=1}^{K} \pi_r^2 \widehat{V}_{dis,r,t}$;
// sub-mechanism $\mathcal{M}_{pub,t}$
for $PG_r \in \boldsymbol{PG}$ **do**
  Compute $n_{pub,r} \leftarrow \lfloor n_r/2 \rfloor - \sum_{\tau=t-\tau_r^*+1}^{t-1} |U_{pub,r,\tau}|$;
  Set $n_{pp,r} \leftarrow \lfloor n_{pub,r}/2 \rfloor$, and $ST_r \leftarrow$ EQ;
  $V_{pub,r,t}^{\mathrm{Pre}} \leftarrow V_{smpl}\left(\lfloor n_r/2 \rfloor, n_{pp,r}\right) + V_{LDP}\left(\epsilon_r', n_{pp,r}, d\right)$;
  Add $V_{pub,r,t}^{\mathrm{Pre}}$ to $\boldsymbol{V}_{pub,t}^{\mathrm{Pre}}$;
Construct Pre inputs according to Table III;
$\boldsymbol{\Lambda}_t^{\mathrm{Pre}} \leftarrow$ PAA-CW (Pre inputs) (Algorithm 4);
Compute publication error $\widetilde{err}^{+}(\widehat{\boldsymbol{F}}_{dis,t}, \boldsymbol{V}_{pub,t}^{\mathrm{Pre}})$ using Equation (10);
if $dis > \widetilde{err}^{+}$ *and* $\min_{r \in [K]} n_{pp,r} \geq u_{min}$ **then**
  for $PG_r \in \boldsymbol{PG}$ **do**
    Sample $U_{pub,r,t}$ of size $n_{pp,r}$ from $U_{A,r}$;
    $U_{A,r} \leftarrow U_{A,r} \backslash U_{pub,r,t}$;
    $\widehat{\boldsymbol{f}}_{pub,t}^{(r)} \leftarrow$ FO($D_{pub,r,t}, \epsilon_r'$) with its unbiased $\widehat{V}_{pub,r,t}$;
    Add $\widehat{\boldsymbol{f}}_{pub,t}^{(r)}$ to $\widehat{\boldsymbol{F}}_{pub,t}$ and $\widehat{V}_{pub,r,t}$ to $\widehat{\boldsymbol{V}}_{pub,t}$;
  if $\mathsf{mode}_{fsn} = \mathrm{Post}$ **then**
    Construct Post inputs according to Table III;
    $\boldsymbol{\Lambda}_t^{\mathrm{Post}} \leftarrow$ PAA-CW (Post inputs) (Algorithm 4);
    Set $\boldsymbol{\Lambda}_t^{(fnl)} \leftarrow \boldsymbol{\Lambda}_t^{\mathrm{Post}}$;
  else
    Set $\boldsymbol{\Lambda}_t^{(fnl)} \leftarrow \boldsymbol{\Lambda}^{\mathrm{Pre}}$;
  Fuse $\widetilde{\boldsymbol{F}}_{pub,t}$ by Equation (2) with $\boldsymbol{\Lambda}_t^{(fnl)}$;
  Aggregate $\widehat{\boldsymbol{f}}_{pub,t} \leftarrow \Pi_\Delta\left(\sum_{r \in [K]} \pi_r \widetilde{\boldsymbol{f}}_{pub,t}^{(r)}\right)$;
else
  Set $\widehat{\boldsymbol{f}}_{pub,t} \leftarrow \widehat{\boldsymbol{f}}_{pub,t-1}$;
for $PG_r \in \boldsymbol{PG}$ **do**
  if $t \geq \tau_r^*$ **then**
    $U_{A,r} \leftarrow U_{A,r} \cup U_{dis,r,t-\tau_r^*+1} \cup U_{pub,r,t-\tau_r^*+1}$;
return $\widehat{\boldsymbol{f}}_{pub,t}$.

Algorithm 6: PLPA$^+$ Stream Estimation

Input: Available user set U_A; privacy requirements $(\boldsymbol{w}, \boldsymbol{\epsilon})$; history $(\widehat{\boldsymbol{f}}_1, \ldots, \widehat{\boldsymbol{f}}_{t-1})$; $\mathsf{mode}_{fsn} \in \{\mathrm{Pre}, \mathrm{Post}\}$

Output: $\widehat{\boldsymbol{f}}_t$

The same as Lines 1-10 in Algorithm 5;

// sub-mechanism $\mathcal{M}_{pub,t}$

Initialize $\boldsymbol{PG}_t^{(A)} \leftarrow \emptyset$;

for $PG_r \in \boldsymbol{PG}$ **do**

  Calculate $t_{N,r} = \frac{\left|U_{pub,r,l_r}\right|}{z^*_{opt,dis,r}} - 1$;

  if $t - l_r \le t_{N,r}$ **then**

    Set $ST_r \leftarrow \mathrm{ND}$;

    Add $\widehat{\boldsymbol{f}}^{(r)}_{pub,l_r}$ to $\widehat{\boldsymbol{F}}_{pub,t}$, $\widehat{V}_{pub,r,l_r}$ to $\boldsymbol{V}^{\mathrm{Pre}}_{pub,t}$ and $\widehat{\boldsymbol{V}}_{pub,t}$;

  else

    Calculate $t_{A,r} = t - l_r - t_{N,r}$;

    Set $n_{pp,r} \leftarrow z^*_{opt,dis,r} \cdot \min(t_{A,r}, \tau^*_r)$;

    $V^{\mathrm{Pre}}_{pub,r,t} \leftarrow V_{smpl}\left(\left\lfloor \frac{n_r}{2} \right\rfloor, n_{pp,r}\right) + V_{LDP}\left(\epsilon'_r, n_{pp,r}, d\right)$;

    Set $ST_r \leftarrow \mathrm{EQ}$;

    Add $V^{\mathrm{Pre}}_{pub,r,t}$ to $\boldsymbol{V}^{\mathrm{Pre}}_{pub,t}$, PG_r to $\boldsymbol{PG}_t^{(A)}$;

Set $\widehat{\boldsymbol{f}}_{pub,t} \leftarrow \widehat{\boldsymbol{f}}_{pub,t-1}$;

if $\boldsymbol{PG}_t^{(A)} \neq \emptyset$ **then**

  Construct Pre inputs according to Table III;

  $\boldsymbol{\Lambda}^{\mathrm{Pre}}_t \leftarrow$ PAA-CW (Pre inputs) (Algorithm 4);

  Compute $\widetilde{err}^{+}(\widehat{\boldsymbol{F}}_{dis,t}, \boldsymbol{V}^{\mathrm{Pre}}_{pub,t})$ using Equation (9);

  if $dis > \widetilde{err}^{+}$ *and* $\min_{PG_r \in \boldsymbol{PG}_t^{(A)}} n_{pp,r} \ge u_{min}$ **then**

    for $PG_r \in \boldsymbol{PG}_t^{(A)}$ **do**

      The same as Lines 21–24 in Algorithm 5;

    The same as Lines 25–32 in Algorithm 5;

Recycle users (The same as Lines 35–37 in Algorithm 5);

return $\widehat{\boldsymbol{f}}_{pub,t}$.

therefore PLPD$^+$ *triggers a synchronous publication for all groups.*

Personalized Local Population Absorption Plus: Partial Group Updates with Historical Reuse. Let l_r denote the latest publication time of group PG_r, and let $\widehat{\boldsymbol{f}}^{(r)}_{pub,l_r}$ and U_{pub,l_r} denote its latest group-level publication and reporting users, respectively. These quantities are maintained as stream state. PLPA$^+$ improves population utilization by updating privacy groups independently. At each time slot, groups whose nullified intervals have not expired reuse their latest publications, whereas active groups absorb the populations accumulated since their previous publications to form new candidate publication populations. By setting auxiliary estimates and variances in Table III, PAA-CW therefore operates on mixed current–historical candidates: Pre uses current dissimilarity estimates for active groups and historical publications for reused groups, while Post replaces the active inputs with their realized publication estimates. Algorithm 6 summarizes the complete procedure.

Example 3. *Continue with the three groups in Example 2. Suppose that at time t, PG_2 remains in its nullified interval, whereas the intervals of PG_1 and PG_3 have expired; hence $PG_t^{(A)} = \{PG_1, PG_3\}$ and $PG_t^{(R)} = \{PG_2\}$, as illustrated in Figure 3. Each active group absorbs the population accumulated since its latest publication and forms a potential publication population $n_{pp,r} = z^*_{opt,dis,r} \cdot \min\{t_{A,r}, \tau^*_r\}$, whereas PG_2 generates no new publication and reuses $\widehat{f}^{(2)}_{pub,l_2}$.*

In Pre, the active groups use their current dissimilarity estimates for PAA-CW. For the reused PG_2, $\widehat{f}^{(2)}_{dis,t}$ represents its current target distribution while $\widehat{f}^{(2)}_{pub,l_2}$ remains the publication candidate; their mismatch is handled by the full risk matrix to account for temporal-staleness bias. If $dis > \widetilde{err}^{+,PA}$ and the active groups satisfy $n_{pp,r} \ge u_{min}$, PG_1 and PG_3 publish new estimates while PG_2 is reused. Pre retains the planned weights, whereas Post recomputes them from the realized publication estimates.

D. Analysis

Time Complexity Analysis. Let d denote the domain size, K the number of privacy groups, and n the total number of users. For each target privacy group PG_H, let $q_H = |\mathcal{C}_H| = |\mathcal{S}_H| + 1$, and define $q_{max} = \max_{H \in [K]} q_H \le K$. Let I_{QP} denote the maximum number of iterations required by the active-set simplex-QP solver.

Theorem V.2. *The per-time-slot time complexity of both* PLPD$^+$ *and* PLPA$^+$ *is $O(n + d\sum_{H=1}^{K} q_H^2 + I_{QP}\sum_{H=1}^{K} q_H^3)$, which is upper bounded by $O(n + dKq_{max}^2 + I_{QP}Kq_{max}^3)$. In the worst case where $q_{max} = K$, the complexity becomes $O(n + dK^3 + I_{QP}K^4)$.*

Proof. Please refer to the details in Appendix IX-D2. □

Privacy Analysis. Theorem V.3 gives the privacy guarantee for PLPD$^+$ and PLPA$^+$.

Theorem V.3. *Both* PLPD$^+$ *and* PLPA$^+$ *satisfy* $(\boldsymbol{w}, \boldsymbol{\epsilon})$*-EPLDP.*

Proof. (sketch) GA-OPS ensures that each user in U_r reports at most once in any τ^*_r consecutive slots with budget $\epsilon'_r = \tilde{\epsilon}_r / \lceil \tilde{w}_r / \tau^*_r \rceil$, and all cross-group operations are post-processing and incur no additional privacy loss. Therefore, both solutions satisfy $(\boldsymbol{w}, \boldsymbol{\epsilon})$-EPLDP. The proof details are shown in Appendix IX-E2. □

Utility Analysis. We analyze PLPD$^+$ and PLPA$^+$ in the Pre mode, where fusion weights are determined before the current publication reports are generated. We assume that dissimilarity-estimation noise is independent of subsequent publication-estimation noise. For each privacy group PG_r, assume that $1 \le s_r < \tau^*_r$ new publications occur in each τ^*_r-length recycling cycle, are uniformly distributed, and satisfy the population lower-bound condition in Algorithm 5; integer rounding is omitted.

Let $\Xi \triangleq \sup_t \max_{r,s} \|\boldsymbol{f}^{(r)}_t - \boldsymbol{f}^{(s)}_t\|_2$ be the maximum cross-group shift. For reused groups, let $\Gamma^2_t = \sum_{r \in \boldsymbol{PG}_t^{(R)}} \|\boldsymbol{f}^{(r)}_{l_r} - \boldsymbol{f}^{(r)}_t\|_2^2$ be the temporal staleness and $V_{hist,t} = \sum_{r \in \boldsymbol{PG}_t^{(R)}} V_{pub,r,l_r}$.

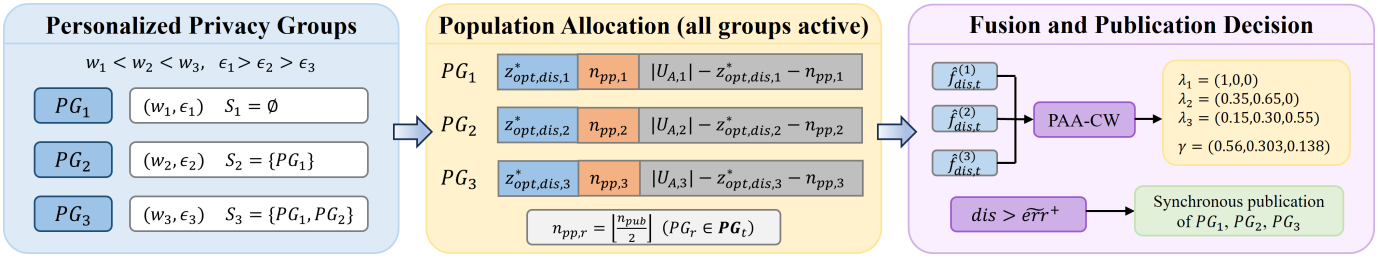


Fig. 2. A PLPD$^+$-Pre example: all groups are active and synchronously updated.

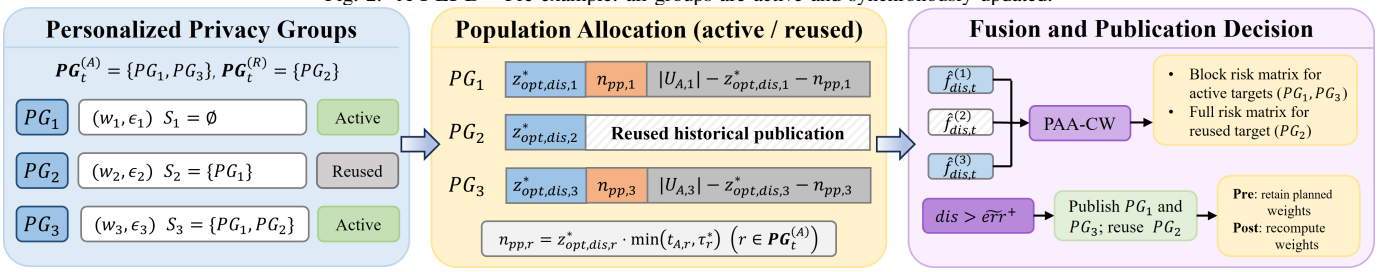


Fig. 3. A PLPA$^+$ example: active groups form candidate publication populations, whereas groups within their nullified intervals reuse historical publications; new reports are generated only if the publication test is satisfied.

Let $\overline{V}_{pub,r} = \frac{2(2^{s_r}-1)}{s_r}\left(\frac{1}{\lfloor n_r/2 \rfloor - 1} + \frac{d-1}{\lfloor n_r/2 \rfloor} \cdot \frac{2e^{\epsilon'_r}+d-2}{(e^{\epsilon'_r}-1)^2}\right) - \frac{1}{\lfloor n_r/2 \rfloor - 1}$, $S_{dis} = \sum_r \overline{V}_{dis,r}$, $D_{dis} = 3R_{grp}\sqrt{d\sum_r \pi_r^2 \overline{V}_{dis,r}} + \sqrt{\sum_r \pi_r^4 Q_r^2}$, where $R_{grp} = \sum_{r=1}^K \pi_r R_r$, $Q_r = \frac{\theta_{dis,r} R_r \sqrt{d\left(1+\overline{V}_{dis,r}\right)}}{1-\theta_{dis,r}}$, $R_r = \frac{e^{\epsilon'_r}+d-2}{e^{\epsilon'_r}-1}$, $\overline{V}_{dis,r} = \widetilde{err}_{FO}\left(\lfloor n_r/2 \rfloor, z^*_{opt,dis,r}, \epsilon'_r\right)$, and $\theta_{dis,r} = \frac{\lfloor n_r/2 \rfloor - z^*_{opt,dis,r}}{z^*_{opt,dis,r}(\lfloor n_r/2 \rfloor - 1)}$.

Theorem V.4. *Assume the historical estimates of reused groups are unbiased. For any slot t with $\boldsymbol{PG}_t^{(A)} \neq \emptyset$ and $\min_{PG_r \in \boldsymbol{PG}_t^{(A)}} n_{pp,r} \geq u_{min}$, PLPA$^+$ in Pre mode satisfies*

$$err_t \leq \sum_{r \in \boldsymbol{PG}_t^{(A)}} V^{\text{Pre}}_{pub,r,t} + \rho_t^2 + 2\sqrt{2}\,\rho_t \sqrt{S_{dis}} + V_{hist,t} + D,$$

where $\rho_t = \Xi + \sqrt{\Gamma_t^2 + V_{hist,t}}$, and $D = 3S_{dis} + \sum_r Q_r + D_{dis}$ collects dissimilarity-estimation terms that vanish as group sizes grow. For PLPD$^+$, all groups are active, thus, $\Gamma_t = V_{hist,t} = 0$, and the time-averaged error satisfies $\overline{err} \leq \sum_r \overline{V}_{pub,r} + \Xi^2 + 2\sqrt{2}\,\Xi\sqrt{S_{dis}} + D$.

Proof. Please refer to the details in Appendix IX-F2. □

Remark. The bounds separate variance from fusion bias: if all privacy groups share the same current distribution, $\Xi = 0$; otherwise the bias is governed by Ξ and Γ_t. They hold for Pre, where the fusion weights are independent of the publication noise; Post is evaluated empirically in Section VI.

VI. Experiments

We conduct experiments on both real-world and synthetic datasets to evaluate the utility and computational efficiency of PLPD$^+$ and PLPA$^+$. We compare them with state-of-the-art non-personalized LDP mechanisms [6], as well as their direct extensions and our PLPD and PLPA. We further conduct ablation studies to evaluate the individual contributions of GA-OPS and BA-PUE, and stress-test the fusion mechanisms under controlled privacy–data correlation.

A. Datasets

Real Datasets. We employ two real-world datasets, namely *Taxi* and *Foursquare*, to evaluate the performance of all competing mechanisms.

Taxi. This dataset records real-time trajectories of $10,357$ taxis in Beijing from February 2 to February 8, 2008. After cleaning, $14,859,377$ valid records from $n = 10,269$ taxis are retained, and locations are mapped into $d = 5$ areas. We sample the trajectories every minute, resulting in $T = 8,889$ time slots.

Foursquare. This dataset consists of $33,278,683$ Foursquare check-ins from $266,909$ users collected between April 2012 and September 2013. We map all venues into $d = 5$ discrete categories. Using 100-minute time slots, we obtain $n = 13,216$ user streams over $T = 7,649$ time slots.

Synthetic Datasets. We generate three synthetic categorical streams with $n = 10{,}000$ users, $T = 10{,}000$ time slots, and $d = 5$. At each time slot t, a generation function produces a probability $p_t \in [0,1]$ for a designated status, and the remaining mass $1 - p_t$ is spread uniformly over the other $d-1$ statuses.

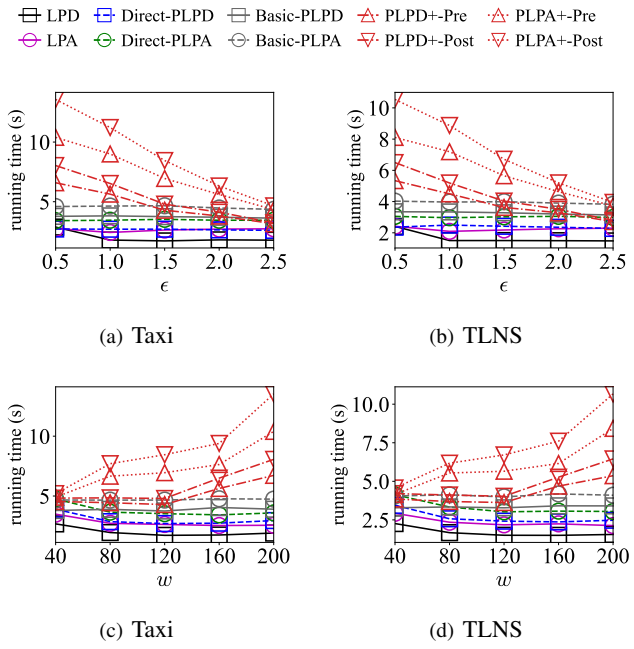
TLNS. The probability evolves according to a clipped random walk: $p_t = \min\{1, \max\{0, p_{t-1} + \eta_t\}\}$, $\eta_t \sim \mathcal{N}(0, Q)$, where $p_0 = 0.05$ and $\sqrt{Q} = 0.0025$.

Sin. The probability follows a sinusoidal pattern: $p_t = A\sin(\omega t) + h$, where $A = 0.05$, $\omega = 0.01$, and $h = 0.5$.

Log. The probability follows a logistic growth pattern: $p_t = \frac{A}{1+e^{-bt}}$, where $A = 0.7$ and $b = 0.02$.

B. Experiment Setup

Compared Algorithms. We compare ten mechanisms. Local Population Distribution (LPD) and Local Population Absorption (LPA) are the non-personalized population-distribution and population-absorption mechanisms, respec-



(a) Taxi (b) TLNS

(c) Taxi (d) TLNS

Fig. 4. The running time with ϵ or w varied.

TABLE IV
EXPERIMENTAL SETTINGS WITH DEFAULT VALUES IN BOLD.

Parameters	Values
global privacy budget ϵ	$0.5, 1.0, \mathbf{1.5}, 2.0, 2.5$
global window size w	$40, 80, \mathbf{120}, 160, 200$
personalized privacy budget ϵ_i	$\epsilon, \ldots, 2.0, 2.5$
personalized window size w_i	$40, 80, \ldots, w$
population lower bound u_{min}	10

tively [6]. Direct-PLPD and Direct-PLPA directly extend these two mechanisms to users with heterogeneous privacy budgets and window sizes, without applying OPS. Basic-PLPD (PLPD) and Basic-PLPA (PLPA) further employ OPS to determine a unified privacy-compatible sampling size. Our enhanced mechanisms, PLPD$^+$ and PLPA$^+$, incorporate both GA-OPS and BA-PUE. For each enhanced mechanism, we evaluate two fusion modes: the Pre mode determines the fusion weights before the current publication estimates are observed, whereas the Post mode recomputes the weights after the current publication estimates become available. Accordingly, the four enhanced variants are denoted by PLPD$^+$-Pre, PLPD$^+$-Post, PLPA$^+$-Pre, and PLPA$^+$-Post.

Parameter Settings. We vary ϵ over $\{0.5, 1.0, 1.5, 2.0, 2.5\}$ and w over $\{40, 80, 120, 160, 200\}$. When varying ϵ, we fix $w = 120$; when varying w, we fix $\epsilon = 1.5$. For personalized mechanisms, ϵ_i is uniformly sampled from candidate values no smaller than ϵ, while w_i is sampled from values no larger than w [7].

All experiments are conducted in Java on an Intel(R) Xeon(R) Silver 4210R CPU at 2.4 GHz with 128 GB of RAM. Each experiment is repeated ten times using different random seeds ranging from 0 to 9, and we report the average results.

Performance Metrics. We evaluate computational efficiency using running time and measure data utility using the average mean squared error (*AMSE*):

$$AMSE = \frac{1}{T}\sum_{\tau=1}^{T}\left\|\hat{\boldsymbol{f}}_\tau - \boldsymbol{f}_\tau\right\|_2^2 = \frac{1}{T}\sum_{\tau=1}^{T}\sum_{j=1}^{d}\left(\hat{f}_{j,\tau} - f_{j,\tau}\right)^2.$$

We also use average Jensen–Shannon divergence (*AJSD*) to measure the data utility. The definition of *AJSD* and the experiment results under *AJSD* are shown in Appendix IX-G2. Lower values indicate better data utility. For readability, the utility figures report $\ln(AMSE)$ and $\ln(AJSD)$.

C. Time Cost Analysis

Figure 4 summarizes the running time under varying ϵ and w. The enhanced variants are slower because GA-OPS and BA-PUE introduce group-specific scheduling and fusion, with Post further recomputing the fusion weights. Increasing ϵ reduces the number of possible privacy groups and therefore lowers the overhead of the enhanced variants. In contrast, increasing w enlarges the candidate set of personalized window sizes and increases the number of groups. The time cost under the other datasets holds the same trend and is shown in Appendix IX-G1.

D. Overall Utility Analysis

Figures 5(a)–5(e) show that $\ln(AMSE)$ generally decreases as ϵ increases, since larger personalized budgets reduce perturbation noise. Across matched variants, population absorption consistently improves over population distribution by accumulating unused publication users, while personalized mechanisms outperform their non-personalized counterparts by exploiting heterogeneous (w_i, ϵ_i). PLPA$^+$-Post further improve over Direct-PLPA by 56.83% (*AMSE*) in average, demonstrating the benefits of GA-OPS and BA-PUE; PLPA$^+$-Post achieves the lowest or nearly lowest error on all five datasets.

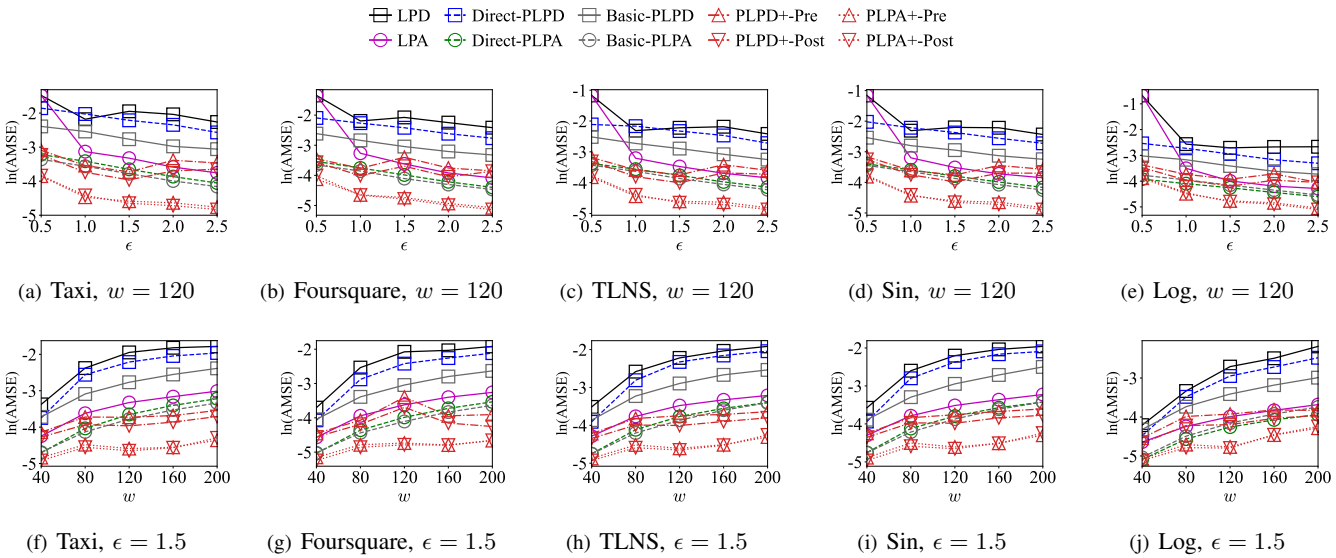
Figures 5(f)–5(j) show the complementary trend with w: larger windows reduce the available reporting population and generally increase $\ln(AMSE)$. PLPA$^+$-Post improve over Direct-PLPA by 51.43% (*AMSE*) in average. The relative ranking of the methods remains consistent with the ϵ-varying results, with PLPA$^+$-Post again providing the best or nearly best utility. The corresponding $\ln(AJSD)$ results are reported in Appendix IX-G2 and exhibit the same trends.

Averaged over all ϵ and w varying settings, PLPA$^+$-Post reduces the AMSE of Direct-PLPA by 52.3% on the two real datasets and by 40.8% on the three synthetic datasets.

E. Ablation Study

To isolate the two components, we compare the PLPD and PLPA with GA-OPS-only variants (group-specific scheduling without fusion), BA-PUE-only variants (bias-aware fusion with OPS), and the complete PLPD$^+$ and PLPA$^+$; the latter two are evaluated in both Pre and Post modes.

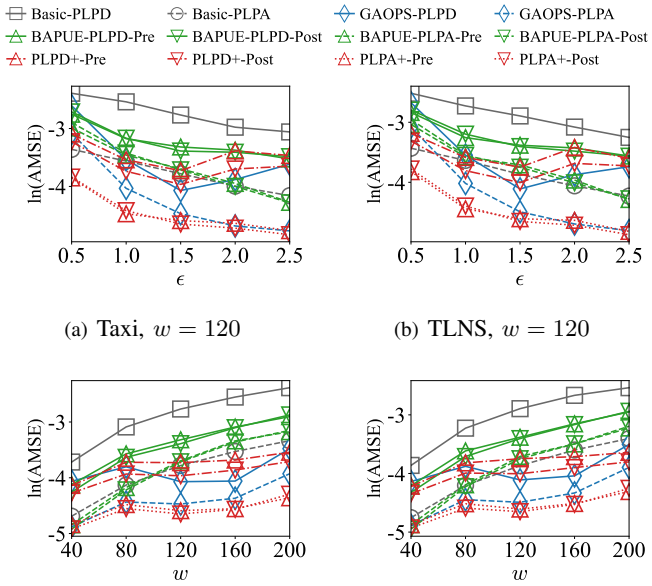
Figure 6 compares the Basic, GA-OPS-only, BA-PUE-only, and complete variants. GA-OPS yields consistent gains, particularly for population absorption, confirming the benefit



(a) Taxi, $w = 120$ (b) Foursquare, $w = 120$ (c) TLNS, $w = 120$ (d) Sin, $w = 120$ (e) Log, $w = 120$

(f) Taxi, $\epsilon = 1.5$ (g) Foursquare, $\epsilon = 1.5$ (h) TLNS, $\epsilon = 1.5$ (i) Sin, $\epsilon = 1.5$ (j) Log, $\epsilon = 1.5$

Fig. 5. *AMSE* with different ϵ or w.



(a) Taxi, $w = 120$ (b) TLNS, $w = 120$

(c) Taxi, $\epsilon = 1.5$ (d) TLNS, $\epsilon = 1.5$

Fig. 6. The ablation study of *AMSE* with different ϵ or w.

of group-specific scheduling. BA-PUE provides additional improvements when variance reduction outweighs distribution-shift bias. Combining both components gives the best overall performance, with Post generally outperforming Pre because its fusion weights use the realized publication estimates. The corresponding $\ln(AJSD)$ results in Appendix IX-G2 show similar trends.

F. Privacy–Data Correlation Analysis

We build a Sin variant (n=120,000, T=1,000, d=5) with four equal-size groups ($\epsilon_i \in \{0.5, 1, 1.5, 2.5\}$, w_i=40), and shift group r's designated-status probability to $p_t + \sigma s_r$ with

TABLE V
AMSE ($\times 10^{-3}$) UNDER POSITIVE PRIVACY–DATA CORRELATION (SIN).

σ	0	0.05	0.10	0.15	0.20
PLPA	1.42	3.39	9.26	19.01	32.67
GA-OPS-PLPA	4.48	4.52	4.45	4.33	4.44
PLPA$^+$-Pre	1.74	2.80	4.78	6.74	7.59
PLPA$^+$-Post	1.89	2.72	4.12	5.71	6.45

$\boldsymbol{s} = (1, \frac{1}{3}, -\frac{1}{3}, -1)$. Since $\sum_r s_r = 0$, varying σ changes only group-level distributions, leaving the population distribution unchanged. As Table V shows, PLPA is the most accurate at $\sigma = 0$, however, degrades by $32.67/1.42 \approx 23$ times at $\sigma = 0.2$; the GA-OPS-PLPA ablation, which never borrows across groups, stays flat. Averaged over the Pre and Post modes, BA-PUE cuts its error by $(4.48/1.89{+}4.48/1.74)/2 \approx 2.5$ times at $\sigma = 0$ while remaining about $(32.67/7.59 + 32.67/6.45)/2 \approx 5$ times better than PLPA at $\sigma = 0.2$. Beyond $\sigma \approx 0.1$, GA-OPS-PLPA becomes preferable to the fusion variants because the distribution-shift bias dominates the variance reduction.

VII. Conclusion

We studied personalized continual frequency estimation under heterogeneous privacy budgets and window sizes, and proposed PLPD and PLPA with OPS and their enhanced variants PLPD$^+$ and PLPA$^+$ with GA-OPS and BA-PUE. Experiments on two real and three synthetic datasets demonstrate their effectiveness and confirm that BA-PUE limits the bias induced by privacy–data correlation.

VIII. AI-Generated Content Acknowledgement

No content of this paper is generated by Generative AI tools and technologies, such as ChatGPT.

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IX. APPENDIX

A. Personalized Frequency Oracle

1) GPRR-based PFO and Unbiasedness: Similar to Frequency Oracle (FO) [34] in the standard LDP model, frequency estimation under PLDP can also be encapsulated into a standard protocol. Here, we refer to this as the Personalized Frequency Oracle (PFO).

PFO consists of three functions: the ***P****erturbation* function PFO.P, the ***A****ggregate* function PFO.A and the ***E****stimation*

function PFO.E. The function PFO.P randomizes the raw data into obfuscated data using the PLDP algorithm. PFO.A calculates the aggregate statistic for different groups of the obfuscated data. PFO.E is responsible for estimating the distribution of raw data.

We adopt Generalized Personalized Random Response (GPRR) as the personalized frequency oracle due to its simplicity and unbiasedness. Each user perturbs his/her value using GPRR with personalized privacy budget ϵ_i:

$$\textbf{PFO.P}: \forall y_i \in \Omega, \Pr[M(x_i) = y_i] = \begin{cases} p_i = \dfrac{e^{\epsilon_i}}{e^{\epsilon_i} + d - 1}, & \text{if } y_i = x_i, \\ q_i = \dfrac{1}{e^{\epsilon_i} + d - 1}, & \text{otherwise.} \end{cases}$$

After receiving the obfuscated reports $\boldsymbol{y}_r$ from all users across different groups divided by distinct $\widetilde{\epsilon}_r \in \widetilde{\boldsymbol{\epsilon}}$, the server computes the obfuscated statistic of each domain value $\omega_j \in \Omega$ for every group G_r:

$$\textbf{PFO.A}: \forall r \in [m], j \in [d], h_{r,j} = \frac{\text{Count}\,(y_{r,j})}{n \cdot g_r},$$

where n is the total number users and $g_r \in \boldsymbol{g}$ is the ratio of users requiring $\widetilde{\epsilon}_r$. Thus, the obfuscated frequency list for each group G_r can be calculated as $\boldsymbol{h}_r = (h_{r,1}, h_{r,2}, \ldots, h_{r,d})$. Subsequently, the server estimates the overall frequency distribution as:

$$\textbf{PFO.E}: \forall \widehat{f}_j \in \widehat{\boldsymbol{f}}, \widehat{f}_j = \sum_{r=1}^{m} \eta_r \cdot \widehat{f}_{r,j} = \sum_{r=1}^{m} \eta_r \cdot \frac{h_{r,j} - \widetilde{q}_r}{\widetilde{p}_r - \widetilde{q}_r}, \tag{11}$$

where $\eta_r = \frac{1/\operatorname{Var}[\widehat{f}_{r,j}]}{\sum_{r'=1}^{m} 1/\operatorname{Var}[\widehat{f}_{r',j}]}$ denotes the *inverse-variance weight* of group G_r, and $\operatorname{Var}\left[\widehat{f}_{r,j}\right]$ represents the variance under ϵ_r-LDP [21] with

$$\operatorname{Var}\left[\widehat{f}_{r,j}\right] = \frac{\widetilde{q}_r(1-\widetilde{q}_r)}{n g_r(\widetilde{p}_r - \widetilde{q}_r)^2} + \frac{1 - \widetilde{p}_r - \widetilde{q}_r}{n g_r(\widetilde{p}_r - \widetilde{q}_r)} \cdot f_j, \tag{12}$$

where $\widehat{p}_r = \frac{e^{\widehat{\epsilon}_r}}{e^{\widehat{\epsilon}_r}+d-1}$ and $\widehat{q}_r = \frac{1}{e^{\widehat{\epsilon}_r}+d-1}$.

Equation (12) shows that the variance depends on the true frequency f_j. However, since f_j is unknown, this definition is impractical for direct computation. To address this issue, we replace $\operatorname{Var}[\cdot]$ with the average variance $\overline{\operatorname{Var}}[\cdot]$, and redefine the weight as $\eta_r = \frac{1/\overline{\operatorname{Var}}[\widehat{f}_{r,j}]}{\sum_{r'=1}^{m} 1/\overline{\operatorname{Var}}[\widehat{f}_{r',j}]}$, where

$$\overline{\operatorname{Var}}\left[\widehat{f}_{r,j}\right] = \frac{\widetilde{q}_r(1-\widetilde{q}_r)}{n g_r(\widetilde{p}_r - \widetilde{q}_r)^2} + \frac{1 - \widetilde{p}_r - \widetilde{q}_r}{n g_r(\widetilde{p}_r - \widetilde{q}_r)} \cdot \frac{1}{d}.$$

Based on the modified weights, we can calculate the total variance in PLDP as:

$$\operatorname{Var}\left[\widehat{\boldsymbol{f}}\right] = 1/\sum_{r=1}^{m} n/\left(d\lambda_r + \mu_r\right), \tag{13}$$

where $\lambda_r = \frac{\widetilde{q}_r(1-\widetilde{q}_r)}{g_r(\widetilde{p}_r-\widetilde{q}_r)^2}$ and $\mu_r = \frac{1-\widetilde{p}_r-\widetilde{q}_r}{g_r(\widetilde{p}_r-\widetilde{q}_r)}$. For more details of the modified weights and the calculation process for the total variance, please refer to Theorem IX.2. We abbreviate the total variance in Equation (13) as $\mathrm{V}_{PFO}(\widetilde{\boldsymbol{\epsilon}}, \boldsymbol{g}, n, d)$.

The inverse-variance aggregation in PFO.E treats the budget-stratified estimates as estimators of a common underlying distribution, which corresponds to the conventional setting where privacy requirements are independent of users' data distributions. When privacy requirements are correlated with data distributions, different privacy groups may represent different underlying distributions, and inverse-variance aggregation no longer preserves the population composition. We address this issue in Section V by estimating privacy groups separately and aggregating them according to their population proportions.

Theorem IX.1. *Suppose that the privacy-budget assignment is independent of users' data values, such that all budget groups share the same underlying distribution $\boldsymbol{f}$. Then GPRR satisfies $\boldsymbol{\epsilon}$-PLDP and $\widehat{f}_j = \sum_{b=1}^{m} \eta_b \cdot \frac{h_{b,j} - \widetilde{q}_b}{\widetilde{p}_b - \widetilde{q}_b}$ is an unbiased aggregation of f_j for all $\omega_j \in \Omega$.*

Proof. For any two input $x_i, x_i' \in \Omega$ of u_i with the output $y_i \in \Omega$, we have

$$\frac{\Pr[M(x_i) = y_i]}{\Pr[M(x_i') = y_i]} \leq \frac{p_i}{q_i} = e^{\epsilon_i}. \tag{14}$$

Thus, GPRR is $\boldsymbol{\epsilon}$-PLDP.

For any expectation $\widehat{f}_j \in \widehat{\boldsymbol{f}}$ of $f_j \in \boldsymbol{f}$, we have

$$\begin{aligned} \mathbb{E}\left[\widehat{f}_j\right] &= \mathbb{E}\left[\sum_{b=1}^{m} \eta_b \cdot \frac{h_{b,j} - \widetilde{q}_b}{\widetilde{p}_b - \widetilde{q}_b}\right] \\ &= \mathbb{E}\left[\sum_{b=1}^{m} \eta_b \cdot \frac{f_j \cdot \widetilde{p}_b + (1 - f_j)\widetilde{q}_b - \widetilde{q}_b}{\widetilde{p}_b - \widetilde{q}_b}\right] \\ &= \mathbb{E}\left[f_j \cdot \sum_{b=1}^{m} \eta_b\right] \\ &= f_j. \end{aligned}$$

Thus, $\widehat{f}_j$ is an unbiased estimation of f_j. □

2) Variance of PFO:

Theorem IX.2. *Under the data-independent privacy-budget assignment assumed in Theorem IX.1, consider the GPRR-based PFO in Equation* (11)*. For each privacy-budget group G_b, let $\widetilde{p}_b$ and $\widetilde{q}_b$ satisfy $\frac{1}{d} < \widetilde{p}_b \leq 1$ and $\widetilde{q}_b = \frac{1-\widetilde{p}_b}{d-1}$. Then the variance of the j-th frequency estimate $\widehat{f}_j \in \widehat{\boldsymbol{f}}$ is*

$$\operatorname{Var}\left[\widehat{f}_j\right] = \begin{cases} V_{A,j} = \dfrac{1}{\sum_{b=1}^{m} n/(\lambda_b + \mu_b \cdot f_j)}, & \text{if } \eta_b = \dfrac{1/\operatorname{Var}[\widehat{f}_{b,j}]}{\sum_{b'=1}^{m} 1/\operatorname{Var}[\widehat{f}_{b',j}]}, \\ V_{B,j} = \dfrac{\sum_{b=1}^{m} \frac{\lambda_b + \mu_b \cdot f_j}{(\lambda_b + \mu_b/d)^2}}{n\left(\sum_{b=1}^{m} 1/(\lambda_b + \mu_b/d)\right)^2}, & \text{if } \eta_b = \dfrac{1/\overline{\operatorname{Var}}[\widehat{f}_{b,j}]}{\sum_{b'=1}^{m} 1/\overline{\operatorname{Var}}[\widehat{f}_{b',j}]}; \end{cases} \tag{15}$$

Moreover, the total variance under the exact inverse-variance weights $\sum_{j=1}^{d} V_{A,j}$ is upper-bounded by the total variance under the average-variance weights $\sum_{j=1}^{d} V_{B,j}$, which satisfies

$$\sum_{j=1}^{d} V_{A,j} \leq \frac{1}{\sum_{b=1}^{m} n/\left(d\lambda_b + \mu_b\right)} = \sum_{j=1}^{d} V_{B,j}. \tag{16}$$

where $\lambda_b = \frac{\widetilde{q}_b(1-\widetilde{q}_b)}{g_b(\widetilde{p}_b-\widetilde{q}_b)^2}$ and $\mu_b = \frac{1-\widetilde{p}_b-\widetilde{q}_b}{g_b(\widetilde{p}_b-\widetilde{q}_b)}$.

Proof. For a fixed privacy-budget group G_b and domain value ω_j, $h_{b,j}$ is the average of ng_b independent Bernoulli random variables. Since the privacy-budget assignment is independent of users' data values, $f_j n g_b$ of these variables have parameter $\widetilde{p}_b$, while $(1-f_j)ng_b$ of them have parameter $\widetilde{q}_b$. Consequently,

$\text{Var}\left[\widehat{f}_{b,j}\right] = (\lambda_b + \mu_b f_j)/n$, and $\overline{\text{Var}}\left[\widehat{f}_{b,j}\right] = (\lambda_b + \mu_b/d)/n$. Thus, if setting $\eta_b = \frac{1/\text{Var}[\widehat{f}_{b,j}]}{\sum_{b'=1}^{m} 1/\text{Var}[\widehat{f}_{b',j}]}$, then

$$\begin{aligned}
\text{Var}\left[\widehat{f}_j\right] =& \sum_{b=1}^{m} \eta_b^2 \cdot \text{Var}\left[\widehat{f}_{b,j}\right] \\
=& \sum_{b=1}^{m} \left(\frac{1/\text{Var}\left[\widehat{f}_{b,j}\right]}{\sum_{i=1}^{m} 1/\text{Var}\left[\widehat{f}_{i,j}\right]} \right)^2 \cdot \text{Var}\left[\widehat{f}_{b,j}\right] \\
=& \frac{\sum_{b=1}^{m} 1/\text{Var}\left[\widehat{f}_{b,j}\right]}{\left(\sum_{i=1}^{m} 1/\text{Var}\left[\widehat{f}_{i,j}\right]\right)^2} \\
=& \frac{1}{\sum_{b=1}^{m} 1/\text{Var}\left[\widehat{f}_{b,j}\right]} \\
=& 1/\sum_{b=1}^{m} n/\left(\lambda_b + \mu_b \cdot f_j\right);
\end{aligned} \tag{17}$$

If setting $\eta_b = \frac{1/\overline{\text{Var}}[\widehat{f}_{b,j}]}{\sum_{b'=1}^{m} 1/\overline{\text{Var}}[\widehat{f}_{b',j}]}$, then

$$\begin{aligned}
\text{Var}\left[\widehat{f}_j\right] =& \sum_{b=1}^{m} \eta_b^2 \cdot \text{Var}\left[\widehat{f}_{b,j}\right] \\
=& \sum_{b=1}^{m} \left(\frac{1/\overline{\text{Var}}\left[\widehat{f}_{b,j}\right]}{\sum_{i=1}^{m} 1/\overline{\text{Var}}\left[\widehat{f}_{i,j}\right]} \right)^2 \cdot \text{Var}\left[\widehat{f}_{b,j}\right] \\
=& \frac{\sum_{b=1}^{m} \text{Var}\left[\widehat{f}_{b,j}\right] / \left(\overline{\text{Var}}\left[\widehat{f}_{b,j}\right]\right)^2}{\left(\sum_{i=1}^{m} 1/\overline{\text{Var}}\left[\widehat{f}_{i,j}\right]\right)^2} \\
=& \frac{1}{n} \cdot \frac{\sum_{b=1}^{m} (\lambda_b + \mu_b f_j) / (\lambda_b + \mu_b/d)^2}{\left(\sum_{b=1}^{m} 1/(\lambda_b + \mu_b/d)\right)^2}.
\end{aligned} \tag{18}$$

For Equation (17), define

$$\phi(x) = \frac{1}{\sum_{b=1}^{m} n/\left(\lambda_b + \mu_b x\right)}.$$

Since $\mu_b \geq 0$ and $\lambda_b + \mu_b x > 0$ for $x \in [0,1]$, $\phi(x)$ is a positive multiple of the harmonic mean of positive affine functions and is therefore concave. Using $\sum_{j=1}^{d} f_j = 1$ and Jensen's inequality, we obtain

$$\begin{aligned}
\sum_{j=1}^{d} V_{A,j} &= \sum_{j=1}^{d} \phi(f_j) \\
&\leq d\phi\left(\frac{1}{d}\right) \\
&= \frac{1}{\sum_{b=1}^{m} n/\left(d\lambda_b + \mu_b\right)}.
\end{aligned}$$

Equality is attained when $f_1 = \cdots = f_d = 1/d$.

For Equation (18), summing over all $j \in [d]$ and using $\sum_{j=1}^{d} f_j = 1$, we obtain

$$\begin{aligned}
\sum_{j=1}^{d} V_{B,j} &= \frac{1}{n} \cdot \frac{\sum_{b=1}^{m} \frac{d\lambda_b + \mu_b}{(\lambda_b + \mu_b/d)^2}}{\left(\sum_{b=1}^{m} \frac{1}{\lambda_b + \mu_b/d}\right)^2} \\
&= \frac{1}{n} \cdot \frac{d \sum_{b=1}^{m} \frac{1}{\lambda_b + \mu_b/d}}{\left(\sum_{b=1}^{m} \frac{1}{\lambda_b + \mu_b/d}\right)^2} \\
&= \frac{1}{\sum_{b=1}^{m} n/\left(d\lambda_b + \mu_b\right)}.
\end{aligned}$$

□

3) Sampling Variance and Reporting Error: **Reporting Errors.** Similar to population division methods under LDP [32], population division under PLDP can also be decomposed into two components: the *variance due to PLDP perturbation* and the *variance due to sampling*. The variance for PLDP perturbation, denoted as $\text{V}_{PFO}(\widetilde{\boldsymbol{\epsilon}}, \boldsymbol{g}, n, d)$, has been shown in Equation (13).

Variance for Sampling. Given a sample of users U with size z, let X_j denote the number of occurrences of $\omega_j \in \Omega$ within this sample. Then X_j follows a Hypergeometric Distribution [35], and its variance can be expressed as

$$\text{Var}_{smpl}\left[X_j\right] = z \cdot f_j \cdot (1 - f_j) \cdot \frac{n - z}{n - 1}.$$

Accordingly, the variance of estimated frequency for Ω_j is:

$$\text{Var}_{smpl}\left[\widehat{f}_j\right] = \frac{1}{z^2} \text{Var}\left[X_j\right] = \frac{f_j \cdot (1 - f_j)}{z} \cdot \frac{n - z}{n - 1}.$$

Thus, the variance of the entire frequency estimation is given by:

$$\text{Var}_{smpl}\left[\widehat{\boldsymbol{f}}\right] = \sum_{j=1}^{d} \frac{f_j \cdot (1 - f_j)}{z} \cdot \frac{n - z}{n - 1} = \frac{n - z}{z(n - 1)} \left(1 - \sum_{j=1}^{d} f_j^2\right). \tag{19}$$

Since $\sum_{j=1}^{d} f_j^2 \geq 0$, Equation (19) is upper bounded by:

$$\text{Var}^{*}_{smpl}\left[\widehat{\boldsymbol{f}}\right] = \frac{n - z}{z(n - 1)}. \tag{20}$$

We abbreviate Equation (20) as $\text{V}_{smpl}\left(n, z\right)$.

Reporting Errors for population division under PLDP. The total variance combining both sampling and perturbation effects is expressed as:

$$\begin{aligned}
&\text{V}(\widetilde{\boldsymbol{\epsilon}}, \boldsymbol{g}, n, z, d) \\
=& \text{V}_{smpl}\left(n, z\right) + \text{V}_{PFO}(\widetilde{\boldsymbol{\epsilon}}, \boldsymbol{g}, z, d) \\
=& \frac{n - z}{z(n - 1)} + 1/\sum_{r=1}^{m} z/\left(d\lambda_r + \mu_r\right),
\end{aligned} \tag{21}$$

We defined the reporting error as the average variance across all values in the output domain, given by:

$$\begin{aligned}
&err(\widetilde{\boldsymbol{\epsilon}}, \boldsymbol{g}, n, z, d) \\
=& \frac{1}{d}\left(\text{V}_{smpl}\left(n, z\right) + \text{V}_{PFO}(\widetilde{\boldsymbol{\epsilon}}, \boldsymbol{g}, z, d)\right) \\
=& \frac{n - z}{dz(n - 1)} + 1/\sum_{r=1}^{m} dz/\left(d\lambda_r + \mu_r\right),
\end{aligned}$$

which coincides with Equation (1).

B. Private Dissimilarity

The ideal method of strategy determination is using non-private dissimilarity dis^* for comparison. The expression of non-private dissimilarity is defined as the mean square error between the true frequency $\boldsymbol{f}_t$ and the latest publication $\widetilde{\boldsymbol{f}}_l$:

$$dis^* = \frac{1}{d}\sum_{j=1}^{d}\left(\boldsymbol{f}_t[j] - \widehat{\boldsymbol{f}}_l[j]\right)^2, \tag{22}$$

where $\widehat{\boldsymbol{f}}_l$ represents the previous frequency publication that consumes population resources. However, dis^* as an intermediate result leads to privacy leakage. To overcome this limitation, we use a private dissimilarity dis instead. The design of dis has two requirements: (1) dis needs to be private, meaning that given a privacy requirement $\boldsymbol{\epsilon}$ for dis calculation, it needs to achieve $\boldsymbol{\epsilon}$-PLDP; (2) dis needs to be an unbiased estimation of the non-private dissimilarity dis^*, ensuring accurate results.

Theorem IX.3. *Let $\widehat{\boldsymbol{f}}_{\text{dis},t}$ denote the unbiased frequency estimation of $\boldsymbol{f}_t$ from an $\boldsymbol{\epsilon}$-PLDP frequency estimation mechanism $\mathcal{M}_{\text{dis},t}$ at current time slot t. Then the following dissimilarity measure*

$$dis = \frac{1}{d}\sum_{j=1}^{d}\left(\widehat{\boldsymbol{f}}_{\text{dis},t}[j] - \widehat{\boldsymbol{f}}_l[j]\right)^2 - \frac{1}{d}\sum_{j=1}^{d}\text{Var}\left[\widehat{\boldsymbol{f}}_{\text{dis},t}[j]\right] \tag{23}$$

is $\boldsymbol{\epsilon}$-PLDP and an unbiased estimation of dis^* in Equation (22).

Proof. We first prove (1) dis in Equation (23) is $\boldsymbol{\epsilon}$-PLDP (*Privacy Proof*) and then prove (2) dis in Equation (23) is an unbiased estimation of dis^* in Equation (22) (*Unbiasedness Proof*).

Privacy Proof. The calculation of dis is only related to $\{y_1, \ldots, y_n\}$ of all users, which is published utilizing GRR. Thus, According to Theorem IX.1, we get dis is $\boldsymbol{\epsilon}$-PLDP.

Unbiasedness Proof. Since $\widehat{\boldsymbol{f}}_{dis,t}$ is an unbiased estimation of $\boldsymbol{f}_t$, for any $\widehat{\boldsymbol{f}}_{dis,t}[j] \in \widehat{\boldsymbol{f}}_{dis,t}$ and $\boldsymbol{f}_t[j] \in \boldsymbol{f}_t$, we have

$$\mathbb{E}\left[\widehat{\boldsymbol{f}}_{dis,t}[j]\right] = \boldsymbol{f}_t[j].$$

Besides, for any random variable X and constant C, according to $\text{Var}[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$, we have

$$\begin{aligned}\text{Var}[X] &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\ &= \mathbb{E}\left[((X - C) - (\mathbb{E}[X] - C))^2\right] \\ &= \mathbb{E}\left[(X - C)^2\right] - (\mathbb{E}[X] - C)^2.\end{aligned}$$

Since last reporting value $\widehat{\boldsymbol{f}}_l$ has been a constant for the current value $\widehat{\boldsymbol{f}}_{dis,t}$, thus, we have

$$\text{Var}\left[\widehat{\boldsymbol{f}}_{dis,t}[j]\right] = \mathbb{E}\left[\left(\widehat{\boldsymbol{f}}_{dis,t}[j] - \widehat{\boldsymbol{f}}_l[j]\right)^2\right] - \left(\boldsymbol{f}_t[j] - \widehat{\boldsymbol{f}}_l[j]\right)^2.$$

Therefore,

$$\mathbb{E}\left[\left(\widehat{\boldsymbol{f}}_{dis,t}[j] - \widehat{\boldsymbol{f}}_l[j]\right)^2\right] = \left(\boldsymbol{f}_t[j] - \widehat{\boldsymbol{f}}_l[j]\right)^2 + \text{Var}\left[\widehat{\boldsymbol{f}}_{dis,t}[j]\right].$$

Hence, the expectation of dis in Equation (23) satisfies

$$\begin{aligned}&\mathbb{E}[dis] \\ =&\mathbb{E}\left[\frac{1}{d}\sum_{j=1}^{d}\left(\widehat{\boldsymbol{f}}_{dis,t}[j] - \widehat{\boldsymbol{f}}_l[j]\right)^2 - \frac{1}{d}\sum_{j=1}^{d}\text{Var}\left[\widehat{\boldsymbol{f}}_{dis,t}\right]\right] \\ =&\frac{1}{d}\sum_{j=1}^{d}\mathbb{E}\left[\left(\widehat{\boldsymbol{f}}_{dis,t}[j] - \widehat{\boldsymbol{f}}_l[j]\right)^2\right] - \frac{1}{d}\sum_{j=1}^{d}\text{Var}\left[\widehat{\boldsymbol{f}}_{dis,t}\right] \\ =&\frac{1}{d}\sum_{j=1}^{d}\left(\left(\boldsymbol{f}_t[j] - \widehat{\boldsymbol{f}}_l[j]\right)^2 + \text{Var}\left[\widehat{\boldsymbol{f}}_{dis,t}[j]\right]\right) - \frac{1}{d}\sum_{j=1}^{d}\text{Var}\left[\widehat{\boldsymbol{f}}_{dis,t}\right] \\ =&\frac{1}{d}\sum_{j=1}^{d}\left(\boldsymbol{f}_t[j] - \widehat{\boldsymbol{f}}_l[j]\right)^2 \\ =&dis^*.\end{aligned}$$

□

C. Analysis of BA-PUE

1) Proof of Theorem V.1:

Proof. By Equation (19), the vector-level variance introduced by sampling z_r users without replacement from a finite population of size N_r is

$$V_{smpl,r} = \theta_r\left(1 - \left\|\boldsymbol{f}^{(r)}\right\|_2^2\right),$$

where $\theta_r = \frac{N_r - z_r}{z_r(N_r - 1)}$. By the law of the total variance, the sampling variance and the conditional GPRR perturbation variance $V_{LDP,r}$ contribute additively to the total vector-level variance. Namely, the total variance of $\widehat{\boldsymbol{f}}^{(r)}$ is

$$V_r = \theta_r\left(1 - \left\|\boldsymbol{f}^{(r)}\right\|_2^2\right) + V_{LDP,r}.$$

Since $\widehat{\boldsymbol{f}}^{(r)}$ is an unbiased estimate of $\boldsymbol{f}^{(r)}$, we have

$$\begin{aligned}\mathbb{E}\left[\left\|\widehat{\boldsymbol{f}}^{(r)}\right\|_2^2\right] &= \sum_{j=1}^{d}\mathbb{E}\left[\widehat{\boldsymbol{f}}^{(r)}[j]^2\right] \\ &= \sum_{j=1}^{d}\left(\boldsymbol{f}^{(r)}[j]^2 + \text{Var}\left[\widehat{\boldsymbol{f}}^{(r)}[j]\right]\right) \\ &= \left\|\boldsymbol{f}^{(r)}\right\|_2^2 + V_r.\end{aligned}$$

Since $2 \le z_r \le N_r$, thus, $0 \le \theta_r < 1$. Taking the expectation of $\widehat{V}_r$ gives

$$\begin{aligned}\mathbb{E}\left[\widehat{V}_r\right] &= \mathbb{E}\left[\frac{\theta_r\left(1 - \left\|\widehat{\boldsymbol{f}}^{(r)}\right\|_2^2\right) + V_{LDP,r}}{1 - \theta_r}\right] \\ &= \frac{\theta_r\left(1 - \mathbb{E}\left[\left\|\widehat{\boldsymbol{f}}^{(r)}\right\|_2^2\right]\right) + V_{LDP,r}}{1 - \theta_r} \\ &= \frac{\theta_r\left(1 - \left\|\boldsymbol{f}^{(r)}\right\|_2^2 - V_r\right) + V_{LDP,r}}{1 - \theta_r} \\ &= \frac{\theta_r\left(1 - \left\|\boldsymbol{f}^{(r)}\right\|_2^2\right) + V_{LDP,r} - \theta_r V_r}{1 - \theta_r} \\ &= \frac{V_r - \theta_r V_r}{1 - \theta_r} \\ &= V_r.\end{aligned}$$

Hence, $\widehat{V}_r$ is an unbiased estimator of the vector-level total variance V_r. □

Algorithm 7: Positive Definite Eigenvalue Clipping (PDEC)

Input: An Empirical Fusion-Error Matrix $\widehat{\boldsymbol{M}}$; the eigenvalue lower bound $\mu > 0$
Output: The PDEC-Corrected Fusion-Error Matrix $\widetilde{\boldsymbol{M}}$

Get the eigenvalue decomposition $\boldsymbol{Q} \cdot \mathrm{diag}(\boldsymbol{\nu}) \cdot \boldsymbol{Q}^\top \leftarrow \widehat{\boldsymbol{M}}$;
for $\nu_k \in \boldsymbol{\nu}$ **do**
  $\widetilde{\nu}_k \leftarrow \max(\nu_k, \mu)$;
Combine the corrected eigenvalues $\widetilde{\boldsymbol{\nu}} \leftarrow (\widetilde{\nu}_1, \ldots, \widetilde{\nu}_{|\boldsymbol{\nu}|})$;
$\widetilde{\boldsymbol{M}} \leftarrow \boldsymbol{Q} \cdot \mathrm{diag}(\widetilde{\boldsymbol{\nu}}) \cdot \boldsymbol{Q}^\top$;
return $\widetilde{\boldsymbol{M}}$;

2) Algorithm for PDEC and Fusion-Error Reduction:

Theorem IX.4 (PDEC-Corrected Fusion-Error Reduction)**.** *For a target privacy group PG_H, let $\widetilde{\boldsymbol{M}}_H = \begin{bmatrix} \widetilde{V}_H & \mathbf{0}^\top \\ \mathbf{0} & \widetilde{\boldsymbol{A}}_H \end{bmatrix}$, where $\widetilde{V}_H = \max\{V_{risk,H}, \mu\}$. Define the minimum PDEC-corrected source fusion error as $\kappa_H = \min_{\substack{\boldsymbol{\beta} \succeq \mathbf{0} \\ \mathbf{1}^\top \boldsymbol{\beta} = 1}} \boldsymbol{\beta}^\top \widetilde{\boldsymbol{A}}_H \boldsymbol{\beta}$. Then, relative to the target-only corrected objective $\widetilde{V}_H$, the minimum BA-PUE fusion objective achieves a reduction of*

$$\widetilde{\Delta}_H = \frac{\widetilde{V}_H^2}{\widetilde{V}_H + \kappa_H}.$$

Proof. Let $\boldsymbol{\lambda}_{\mathcal{S}_H}$ denote the vector fusion weights assigned to the source groups in $\mathcal{S}_H$, and let $\alpha_H = \mathbf{1}^\top \boldsymbol{\lambda}_{\mathcal{S}_H} = 1 - \lambda_H$ denote the total source weight. For $\alpha_H > 0$, define the normalized source-weight vector as $\boldsymbol{\beta} = \frac{\boldsymbol{\lambda}_{\mathcal{S}_H}}{\alpha_H}$, where $\boldsymbol{\beta} \succeq \mathbf{0}$ and $\mathbf{1}^\top \boldsymbol{\beta} = 1$. The case $\alpha_H = 0$ corresponds to using the target estimate alone and is included as a boundary case. Define the minimum PDEC-corrected source risk as

$$\kappa_H = \min_{\substack{\boldsymbol{\beta} \succeq \mathbf{0} \\ \mathbf{1}^\top \boldsymbol{\beta} = 1}} \boldsymbol{\beta}^\top \widetilde{\boldsymbol{A}}_H \boldsymbol{\beta}.$$

Under the current-estimate specialization, the PDEC-corrected risk matrix is $\widetilde{\boldsymbol{M}}_H = \begin{bmatrix} \widetilde{V}_H & \mathbf{0}^\top \\ \mathbf{0} & \widetilde{\boldsymbol{A}}_H \end{bmatrix}$, where $\widetilde{V}_H = \max\{V_{risk,H}, \mu\}$. Then we have

$$\widetilde{R}_H(\alpha_H, \boldsymbol{\beta}) = \widetilde{V}_H(1 - \alpha_H)^2 + \alpha_H^2 \boldsymbol{\beta}^\top \widetilde{\boldsymbol{A}}_H \boldsymbol{\beta}. \tag{24}$$

Let $\boldsymbol{\beta}^* = \arg\min_{\substack{\boldsymbol{\beta} \succeq \mathbf{0} \\ \mathbf{1}^\top \boldsymbol{\beta} = 1}} \boldsymbol{\beta}^\top \widetilde{\boldsymbol{A}}_H \boldsymbol{\beta}$. By the definition of κ_H, minimizing $\widetilde{R}_H$ is therefore reduced to

$$\min_{0 \le \alpha_H \le 1} \widetilde{V}_H(1 - \alpha_H)^2 + \alpha_H^2 \kappa_H.$$

Its derivative is

$$\frac{\partial \widetilde{R}_H}{\partial \alpha_H} = -2\widetilde{V}_H(1 - \alpha_H) + 2\alpha_H \kappa_H.$$

By setting

$$-2\widetilde{V}_H(1 - \alpha_H) + 2\alpha_H \kappa_H = 0,$$

we have

$$\alpha_H^* = \frac{\widetilde{V}_H}{\widetilde{V}_H + \kappa_H},$$
$$\lambda_H^* = 1 - \alpha_H^* = \frac{\kappa_H}{\widetilde{V}_H + \kappa_H},$$
$$\widetilde{R}_H^* = \frac{\widetilde{V}_H \kappa_H}{\widetilde{V}_H + \kappa_H}.$$

Therefore,

$$\widetilde{\Delta}_H = \widetilde{V}_H - \widetilde{R}_H^* = \frac{\widetilde{V}_H^2}{\widetilde{V}_H + \kappa_H}.$$

□

When κ_H is substantially larger than $\widetilde{V}_H$, the optimal total weight assigned to the source groups approaches zero and $\widetilde{\Delta}_H$ approaches zero. BA-PUE therefore naturally falls back to the target-group estimate when the available source groups provide little transferable utility.

D. Proofs of Theorems for Time Complexity

1) Proof of Theorem IV.1:

Proof. We analyze the computational complexity of PLPD and PLPA at an arbitrary time slot t.

Note that OPS is executed only once during initialization. Specifically, OPS depends on the users' privacy-window requirements $\boldsymbol{w}$, privacy budgets $\boldsymbol{\epsilon}$, the population size n, and the domain size d, all of which remain unchanged during stream processing. Therefore, the optimized sampling size z_{opt}, recycling period τ_{opt}, and effective privacy-budget vector $\boldsymbol{\epsilon}'_{opt}$ are computed once and reused at all subsequent time slots. Hence, the one-time cost of OPS is not included in the per-time-slot complexity analyzed below.

In addition, the remaining population size and recycling status related to τ_{opt} are maintained incrementally during stream processing. Therefore, checking the availability of users and updating the recycling population incurs $O(1)$ time per expired or newly generated population, rather than scanning the entire τ_{opt}-length history.

At each time slot, PLPD and PLPA consist of two sub-mechanisms, namely the dissimilarity mechanism $\mathcal{M}_{dis,t}$ and the publication mechanism $\mathcal{M}_{pub,t}$.

For $\mathcal{M}_{dis,t}$, a user subset of size $z_{opt} \le n$ is sampled from the available population. Each selected user applies GPRR once, and the resulting report is inserted into the counter corresponding to its effective privacy budget and reported domain value. Since each user belongs to exactly one effective-budget group, perturbing and aggregating all selected reports require $O(z_{opt})$ time cost.

After aggregation, PFO.E reconstructs the frequency distribution. For each of the d domain values, it combines the estimates from the m_{eff} effective-budget groups. Therefore, this step requires $O(d \cdot m_{eff})$ time cost. Computing the dissimilarity between the current private estimate and the last publication requires an additional $O(d)$ time cost. Consequently,

$$O(\mathcal{M}_{dis,t}) = O\left(z_{opt} + d \cdot m_{eff} + d\right) = O\left(n + d \cdot m_{eff}\right).$$

For $\mathcal{M}_{pub,t}$, PLPD and PLPA differ in how the publication population is allocated: PLPD distributes the remaining population, whereas PLPA determines the publication population through population absorption. Nevertheless, these population-management operations involve at most $O(n)$ users at a time slot and therefore do not exceed $O(n)$ time cost.

If a new publication is triggered, at most n users can participate in $\mathcal{M}_{pub,t}$. As in $\mathcal{M}_{dis,t}$, perturbing and aggregating their reports require $O(n)$ time, while the subsequent PFO estimation over the d domain values and m_{eff} effective-budget groups requires $O(d \cdot m_{eff})$ time cost. Hence, $O(\mathcal{M}_{pub,t}) = O(n + d \cdot m_{eff})$. If no new publication is triggered, the previous publication is directly reused, whose cost is no larger than the above bound.

Therefore, for either PLPD or PLPA, the total computational cost at an arbitrary time slot t is $O(n + d \cdot m_{eff})$. □

2) Proof of Theorem V.2:

Proof. We analyze the computational complexity of PLPD^+ and PLPA^+ at an arbitrary time slot t.

Similar to OPS in PLPD and PLPA, GA-OPS depends only on the privacy-group information, including the group population sizes, privacy-window sizes, privacy budgets, and the domain size, all of which remain unchanged during stream processing. The candidate source sets $\{\mathcal{S}_H \mid H \in [K]\}$ also depend only on the privacy requirements. Therefore, GA-OPS and the candidate source sets can be computed once during initialization and reused at all subsequent time slots. Their one-time initialization costs are thus not included in the following per-time-slot complexity.

We first analyze the dissimilarity mechanism $\mathcal{M}_{dis,t}$. Since the K privacy groups are disjoint, the total number of sampled users over all groups is at most n. Hence, sampling, perturbing, and aggregating the reports require $O(n)$ time cost. For each privacy group, constructing its d-dimensional frequency estimate and computing its variance estimate require $O(d)$ time cost. Therefore, these operations over all K privacy groups require $O(dK)$ time. The population-weighted aggregation and the subsequent dissimilarity computation also require at most $O(dK)$ time. Thus,

$$O\left(\mathcal{M}_{dis,t}\right) = O\left(n + dK\right).$$

We next analyze the fusion-weight computation in the publication mechanism $\mathcal{M}_{pub,t}$. For each target privacy group PG_H, let $\mathcal{C}_H = \{H\} \cup \mathcal{S}_H$ and $q_H = |\mathcal{C}_H| = |\mathcal{S}_H| + 1$.

For PAA-CW in PLPD^+, the empirical source-risk block contains at most q_H^2 entries. Similarly, for PAA-CW in PLPA^+, either a full empirical risk matrix or a block empirical risk matrix is constructed, both of which contain at most q_H^2 entries. Since each entry involves an inner product between two d-dimensional distribution-shift vectors, constructing the empirical risk matrix for PG_H requires $O(dq_H^2)$ time cost.

After the empirical risk matrix is constructed, PDEC performs eigenvalue decomposition on a matrix of size at most $q_H \times q_H$, which requires $O\left(q_H^3\right)$ time. The corrected fusion weights are then obtained by the active-set simplex-QP solver. Let $I_{QP} \geq 1$ denote the maximum number of active-set iterations. Each iteration involves solving a dense linear system of dimension at most q_H, and therefore requires at most $O\left(q_H^3\right)$ time cost. Consequently, the fusion-weight computation for target group PG_H requires

$$O\left(d \cdot q_H^2 + I_{QP} \cdot q_H^3\right),$$

where the $O\left(q_H^3\right)$ cost of PDEC is absorbed into the second term because $I_{QP} \geq 1$.

Summing over all target privacy groups, the total cost of computing the fusion weights is

$$O\left(d \sum_{H=1}^{K} q_H^2 + I_{QP} \sum_{H=1}^{K} q_H^3\right).$$

After obtaining the fusion weights, BA-PUE combines at most q_H d-dimensional candidate estimates for each target privacy group. Therefore, the total fusion cost is

$$O\left(d \sum_{H=1}^{K} q_H\right),$$

which is dominated by $O\left(d \sum_{H=1}^{K} q_H^2\right)$. The potential publication error in Equation (10) or Equation (9) can also be evaluated within this bound. The projection in Line 32 of Algorithm 5 takes $O(d)$ expected time, which is dominated by $O(d \sum_H q_H^2)$.

If a new publication is triggered, sampling, perturbing, and aggregating the publication users over all privacy groups require at most $O(n)$ time cost, while constructing the group-level frequency estimates and variance estimates requires $O(dK)$ time cost. In POST mode, PAA-CW is invoked once again after the actual publication estimates become available. This only introduces a constant factor and therefore does not change the asymptotic complexity. The population-availability and recycling statistics can also be maintained incrementally, as in PLPD and PLPA.

Consequently, the per-time-slot time complexity of both PLPD^+ and PLPA^+ is

$$O\left(n + d \sum_{H=1}^{K} q_H^2 + I_{QP} \sum_{H=1}^{K} q_H^3\right).$$

Let $q_{max} = \max_{H \in [K]} q_H$. Since $q_H \leq q_{max} \leq K$, we have $\sum_{H=1}^{K} q_H^2 \leq K q_{max}^2$, and $\sum_{H=1}^{K} q_H^3 \leq K q_{max}^3$. Therefore, the complexity is upper bounded by

$$O\left(n + dK q_{max}^2 + I_{QP} K q_{max}^3\right).$$

In the worst case where $q_{max} = K$, this becomes

$$O\left(n + dK^3 + I_{QP} K^4\right).$$

□

E. Proofs of Theorems for Privacy Analysis

According to relationship between the definitions of $(\boldsymbol{w}, \boldsymbol{\epsilon})$-EPLDP, $\boldsymbol{\epsilon}$-PLDP, we have the following important three lemmas for privacy analysis.

Lemma IX.1. *Given any window size $w \in \mathbb{N}^+$, a mechanism $\mathcal{M}$ satisfies $(w \cdot \mathbf{1}, \boldsymbol{\epsilon})$-EPLDP if it satisfies $\boldsymbol{\epsilon}$-PLDP in any window of size w.*

Proof. Let S_t and $\widehat{S}_t$ be any w-neighboring stream prefixes. If $S_t = \widehat{S}_t$, then for any user $u_i \in \mathcal{U}$, it is obvious that $\Pr[\mathcal{M}(S_t)] \leq e^{\boldsymbol{\epsilon}(u_i)} \cdot \Pr\left[\mathcal{M}\left(\widehat{S}_t\right)\right]$. Otherwise, we can find $\tau_1, \tau_2 \in [t]$ satisfying $\tau_2 - \tau_1 + 1 \leq w$, with $S_t[\tau_1] \neq \widehat{S}_t[\tau_1]$ and $S_t[\tau_2] \neq \widehat{S}_t[\tau_2]$, meanwhile for any $\tau \in [1, \tau_1 - 1] \cup [\tau_2 + 1, t]$ it satisfies $S_t[\tau] = \widehat{S}_t[\tau]$, and for any $\tau \in [\tau_1, \tau_2]$, it satisfies $S_t[\tau]$ and $\widehat{S}_t[\tau]$ are neighboring.

Let $\widetilde{\boldsymbol{\epsilon}} = \text{Unique}(\boldsymbol{\epsilon})$ be the unique ϵ among all users. Let m be the number of $\widetilde{\boldsymbol{\epsilon}}$. For any two w length sub-stream segments $S_{\tau-w+1,\tau}$ and $\widehat{S}_{\tau-w+1,\tau}$ satisfying $\tau - w + 1 \leq \tau_1 \leq \tau_2 \leq \tau$, since $\mathcal{M}$ satisfies $\boldsymbol{\epsilon}$-PLDP within any w window, then for any personalized subset-stream segment $S^{(k)}_{\tau-w+1,\tau} \subseteq S_{\tau-w+1,\tau}$ and $\widehat{S}^{(k)}_{\tau-w+1,\tau} \subseteq S_{\tau-w+1,\tau}$, we have $\forall y \in \text{Range}(\mathcal{M})$,

$$\Pr\left[\mathcal{M}\left(S^{(k)}_{\tau-w+1,\tau}\right) = y\right] \leq e^{\boldsymbol{\epsilon}(u_i)} \Pr\left[\mathcal{M}\left(\widehat{S}^{(k)}_{\tau-w+1,\tau}\right) = y\right].$$

Therefore, for any personalized stream Prefix $S^{(k)}_t \subseteq S_t, \widehat{S}^{(k)}_t \subseteq \widehat{S}_t$ and $\forall y \in \text{Range}(\mathcal{M})$ with $y = y_{1:\tau-w} \| y_{\tau-w+1,\tau} \| y_{\tau+1,t}$, we have

$$\begin{aligned}
&\frac{\Pr\left[\mathcal{M}\left(S^{(k)}_t\right) = y\right]}{\Pr\left[\mathcal{M}\left(\widehat{S}^{(k)}_t\right) = y\right]} \\
\leq &\frac{\Pr\left[\mathcal{M}\left(S^{(k)}_{1,\tau-w}\right) = y_{1,\tau-w}\right]}{\Pr\left[\mathcal{M}\left(\widehat{S}^{(k)}_{1,\tau-w}\right) = y_{1,\tau-w}\right]} \times \frac{\Pr\left[\mathcal{M}\left(S^{(k)}_{\tau-w+1,\tau}\right) = y_{\tau-w+1,\tau}\right]}{\Pr\left[\mathcal{M}\left(\widehat{S}^{(k)}_{\tau-w+1,\tau}\right) = y_{\tau-w+1,\tau}\right]} \\
&\times \frac{\Pr\left[\mathcal{M}\left(S^{(k)}_{\tau+1,t}\right) = y_{\tau+1,t}\right]}{\Pr\left[\mathcal{M}\left(\widehat{S}^{(k)}_{\tau+1,t}\right) = y_{\tau+1,t}\right]} \\
= &\frac{\Pr\left[\mathcal{M}\left(S^{(k)}_{\tau-w+1,\tau}\right) = y_{\tau-w+1,\tau}\right]}{\Pr\left[\mathcal{M}\left(\widehat{S}^{(k)}_{\tau-w+1,\tau}\right) = y_{\tau-w+1,\tau}\right]} \\
\leq &e^{\boldsymbol{\epsilon}(u_i)}.
\end{aligned}$$

Therefore, $\mathcal{M}$ satisfies $(w \cdot \mathbf{1}, \boldsymbol{\epsilon})$-EPLDP. □

Lemma IX.2. *Given $\widehat{w} \in \mathbb{N}^+$, $\mathcal{M}$ satisfies $(\boldsymbol{w}, \boldsymbol{\epsilon})$-EPLDP if it satisfies $(\widehat{w} \cdot \mathbf{1}, \widehat{\boldsymbol{\epsilon}})$-EPLDP where*

$$\widehat{\boldsymbol{\epsilon}}(u_i) = \begin{cases} \boldsymbol{\epsilon}(u_i), & \text{if } \boldsymbol{w}(u_i) \leq \widehat{w}, \\ \boldsymbol{\epsilon}(u_i) / \lceil \boldsymbol{w}(u_i)/\widehat{w} \rceil, & \text{otherwise.} \end{cases}$$

Proof. All users can be classified into two cases: (1) the users whose window sizes are no more than $\widehat{w}$; (2) the users whose window sizes are larger than $\widehat{w}$.

For any user u_i in case (1), it satisfies $\boldsymbol{w}(u_i) = w_k \leq \widehat{w}$. Then $\forall S^{(k)}_t, \widehat{S}^{(k)}_t$ with $S^{(k)}_t \sim_{w_k} \widehat{S}^{(k)}_t$, we have $S^{(k)}_t \sim_{\widehat{w}} \widehat{S}^{(k)}_t$. Because $\mathcal{M}$ satisfies $(\widehat{w} \cdot \mathbf{1}, \widehat{\boldsymbol{\epsilon}})$-EPLDP , we have $\forall y \in \text{Range}(\mathcal{M})$,

$$\frac{\Pr\left[\mathcal{M}\left(S^{(k)}_t\right) = y\right]}{\Pr\left[\mathcal{M}\left(\widehat{S}^{(k)}_t\right) = y\right]} \leq e^{\widehat{\boldsymbol{\epsilon}}(u_i)} = e^{\boldsymbol{\epsilon}(u_i)}.$$

For any user u_i in case (2), it satisfies $\boldsymbol{w}(u_i) = w_k > \widehat{w}$. Let $S^{(k)}_t, \widehat{S}^{(k)}_t$ be any two personalized stream prefixes satisfying $S_t \sim_{w_k} \widehat{S}_t$. If $S^{(k)}_t = \widehat{S}^{(k)}_t$, then it is obvious that $\Pr\left[\mathcal{M}\left(S^{(k)}_t\right) = y\right] / \Pr\left[\mathcal{M}\left(\widehat{S}^{(k)}_t\right) = y\right] = 1 \leq e^{\boldsymbol{\epsilon}(u_i)}$. Otherwise, we can find τ_1, τ_2 satisfying $\tau_2 - \tau_1 + 1 \leq w_k, S^{(k)}_t[\tau_1] \neq \widehat{S}^{(k)}_t[\tau_1], S^{(k)}_t[\tau_2] \neq \widehat{S}^{(k)}_t[\tau_2]$ and $\forall \tau \in [1, \tau_1 - 1] \cup [\tau_2 + 1, t]$, $S^{(k)}_t[\tau] = \widehat{S}^{(k)}_t[\tau]$. Let $z = \lceil w_k/\widehat{w} \rceil$, then we can construct a list of $z - 1$ stream prefixes $\check{\boldsymbol{S}} = \left\langle \check{S}^{(k,1)}_t, \check{S}^{(k,2)}_t, \ldots, \check{S}^{(k,z-1)}_t \right\rangle$ satisfying $S^{(k)}_t \sim_{\widehat{w}} \check{S}^{(k,1)}_t$, $\check{S}^{(k,z-1)}_t \sim_{\widehat{w}} \widehat{S}^{(k)}_t$ and $\forall j \in [2, z-1]$, $\check{S}^{(k,j-1)}_t \sim_{\widehat{w}} \check{S}^{(k,j)}_t$. By adding $S^{(k)}_t$ to the head of $\check{\boldsymbol{S}}$ and $\widehat{S}^{(k)}_t$ to tail, we can get the extended list of $z + 1$ stream prefixes $\check{\boldsymbol{S}}^+ = \left\langle S^{(k)}_t, \check{S}^{(k,1)}_t, \check{S}^{(k,2)}_t, \ldots, \check{S}^{(k,z-1)}_t, \widehat{S}^{(k)}_t \right\rangle$. For any neighboring pair prefixes x, x' in $\check{\boldsymbol{S}}^+$, because $\mathcal{M}$ satisfies $(\widehat{w} \cdot \mathbf{1}, \widehat{\boldsymbol{\epsilon}})$-EPLDP, we have $\forall y \in \text{Range}(\mathcal{M}), \Pr[\mathcal{M}(x) = y] / \Pr[\mathcal{M}(x') = y] \leq e^{\epsilon_k/z}$. Therefore, for $\forall y \in \text{Range}(\mathcal{M})$, we have

$$\begin{aligned}
&\frac{\Pr\left[\mathcal{M}\left(S^{(k)}_t\right) = y\right]}{\Pr\left[\mathcal{M}\left(\widehat{S}^{(k)}_t\right) = y\right]} \\
= &\frac{\Pr\left[\mathcal{M}\left(S^{(k)}_t\right) = y\right]}{\Pr\left[\mathcal{M}\left(\check{S}^{(k,1)}_t\right) = y\right]} \times \prod_{j=2}^{z-1} \frac{\Pr\left[\mathcal{M}\left(\check{S}^{(k,j-1)}_t\right) = y\right]}{\Pr\left[\mathcal{M}\left(\check{S}^{(k,j)}_t\right) = y\right]} \\
&\times \frac{\Pr\left[\mathcal{M}\left(\check{S}^{(k,z-1)}_t\right) = y\right]}{\Pr\left[\mathcal{M}\left(\widehat{S}^{(k)}_t\right) = y\right]} \\
\leq &e^{\epsilon_k/z} \times \left(e^{\epsilon_k/z}\right)^{z-2} \times e^{\epsilon_k/z} \\
= &e^{\epsilon_k} \\
= &e^{\boldsymbol{\epsilon}(u_i)}.
\end{aligned}$$

Therefore, $\mathcal{M}$ satisfies $(\boldsymbol{w}, \boldsymbol{\epsilon})$-EPLDP. □

Lemma IX.3. *Let the user set be partitioned into K disjoint privacy groups $U_1, \ldots, U_K$, where all users in $PG_r = (\widetilde{w}_r, \widetilde{\epsilon}_r, U_r)$ share the same privacy requirement. Suppose that, for each PG_r, every user participates at most once within any τ_r consecutive time slots and each participating user applies an ϵ'_r-LDP local randomizer, where*

$$\epsilon'_r = \frac{\widetilde{\epsilon}_r}{\lceil \widetilde{w}_r/\tau_r \rceil}.$$

If all subsequent operations only process the locally private outputs, then the overall mechanism satisfies $(\boldsymbol{w}, \boldsymbol{\epsilon})$-EPLDP.

Proof. Fix an arbitrary privacy group $PG_r = (\widetilde{w}_r, \widetilde{\epsilon}_r, U_r)$. Since each user in U_r participates at most once within any τ_r consecutive time slots and each report is generated by an ϵ'_r-LDP local randomizer, the joint private output within such a window satisfies $\epsilon'_r \cdot \mathbf{1}$-PLDP over U_r.

According to Lemma IX.1, the mechanism restricted to U_r therefore satisfies $(\tau_r \cdot \mathbf{1}, \epsilon'_r \cdot \mathbf{1})$-EPLDP.

Since $\epsilon'_r = \widetilde{\epsilon}_r / \lceil \widetilde{w}_r/\tau_r \rceil$, applying Lemma IX.2 to the user subset U_r with $\widehat{w} = \tau_r$ gives $(\widetilde{w}_r \cdot \mathbf{1}, \widetilde{\epsilon}_r \cdot \mathbf{1})$-EPLDP.

Now fix any user $u_i \in U_r$. The remaining privacy groups contain no raw data of u_i. Moreover, after the locally private group-level reports are obtained, all cross-group operations, including PAA-CW, PDEC, BA-PUE, aggregation, reuse, and projection, operate only on private outputs. Hence, with respect to u_i, conditioned on the data and locally private outputs of all other users, the final mechanism depends on u_i's private stream only through $(\widetilde{w}_r, \widetilde{\epsilon}_r)$-EPLDP output generated from

PG_r. Hence, the subsequent cross-group operations preserve the same privacy bound for u_i.

Since every user belongs to exactly one privacy group, the above argument holds for every $u_i \in U$. Thus, the overall mechanism satisfies $(\boldsymbol{w}, \boldsymbol{\epsilon})$-EPLDP. □

Next, we give a lemma to the privacy analysis of PLPD and PLPA.

Lemma IX.4. *For both* PLPD *and* PLPA*, after applying OPS, each user participates at most once within any τ_{opt}-length consecutive time slots, where τ_{opt} is the recycling period returned by OPS.*

Proof. We only need to prove that the total number of reporting users within any τ_{opt} consecutive time slots is no more than n, i.e., $\sum_{\tau=t-\tau_{opt}+1}^{t} |U_\tau| \le n$.

Since the available-user set is updated by removing selected users after each report and expired users are recycled only after τ_{opt} time slots, it is sufficient to show that the total number of assigned reporting users within any τ_{opt}-length window does not exceed n.

For PLPD, in $\mathcal{M}_{dis,t}$, z_{opt} users are allocated at each time slot. According to the definition of τ_{opt} in OPS with $\rho = 1/2$, $\tau_{opt} z_{opt} \le \frac{n}{2}$. Therefore, within any τ_{opt} consecutive time slots, the total number of users participating in $\mathcal{M}_{dis,t}$ is at most $n/2$. In $\mathcal{M}_{pub,t}$, PLPD either publishes with additional users $U_{pub,t}$ or approximates the last release without any user assignment. In the former case, there are at most $|U_{pub,t}| \le \left(n/2 - \sum_{\tau=t-\tau_{opt+1}}^{t-1} |U_{pub,\tau}|\right)/2$ users. Thus, we have $0 \le \sum_{\tau=t-\tau_{opt}+1}^{t} |U_{pub,\tau}| \le n/2$. In the latter case, $|U_{pub,t}| = 0$. Therefore, for any time slot t, the total number of reporting users within any τ_{opt} consecutive time slots for PLPD satisfies

$$\sum_{\tau=t-\tau_{opt}+1}^{t} |U_\tau| = \sum_{\tau=t-\tau_{opt}+1}^{t} \left|U_{dis,\tau}\right| + \sum_{\tau=t-\tau_{opt}+1}^{t} \left|U_{pub,\tau}\right| \le \frac{n}{2} + \frac{n}{2} = n.$$

For PLPA, similarly, in $\mathcal{M}_{dis,t}$, there are $\sum_{\tau=t-\tau_{opt}+1}^{t} |U_{dis,\tau}| \le n/2$ users for any t and $\tau \in [t]$. In $\mathcal{M}_{pub,t}$, assume there are c new publications in any τ_{opt} consecutive time slots. We denote these publication time slots as $(\tau_1, \tau_2, \ldots, \tau_c)$. For any publication time slot τ_j $(j \in [c])$, we denote the number of its absorbed population share as η_j, thus after absorption, time slot τ_j will occupy $\eta_j + 1$ population shares. If $c = 0$, no publication users are assigned, and hence the total publication population is 0. If $c = 1$, the single publication consumes at most τ_{opt} population shares. Therefore,

$$\sum_{\tau=t-\tau_{opt}+1}^{t} \left|U_{pub,\tau}\right| \le \tau_{opt} z_{opt} \le \frac{n}{2}.$$

It remains to consider the case $c \ge 2$. Figure 7 illustrates an example for $c = 3$ and $\tau_{opt} = 9$, in which, for instance, time slot 3 absorbs population shares from time slots 1 and 2, while nullifying the population at time slots 4 and 5. For $c \ge 2$, the absorbed and nullified time-slot lengths are symmetric with respect to each new publication. Therefore,

$$\tau_{opt} \ge \sum_{j=1}^{c} (1 + 2\eta_j) - \eta_1 - \eta_c.$$

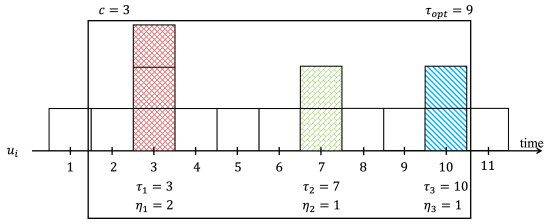


Fig. 7. An example for new publications' absorption in PLPA.

For $c \ge 2$ and any time slot t, we have

$$\begin{aligned}
\sum_{\tau=t-\tau_{opt}+1}^{t} \left|U_{pub,\tau}\right| &\le z_{opt} \cdot \sum_{j=1}^{c} (1+\eta_j) \\
&\le \frac{n}{2\tau_{opt}} \cdot \sum_{j=1}^{c} (1+\eta_j) \\
&\le \frac{n \cdot \sum_{j=1}^{c} (1+\eta_j)}{2\sum_{j=1}^{c} (1+2\eta_j) - 2\eta_1 - 2\eta_c} \\
&= \frac{n \cdot \sum_{j=1}^{c} (1+\eta_j)}{2\sum_{j=1}^{c} (1+\eta_j) + 2\sum_{j=2}^{c-1} \eta_j} \\
&\le n/2.
\end{aligned}$$

Combining the cases $c = 0$, $c = 1$, and $c \ge 2$, the total number of publication users within any τ_{opt}-length window is at most $n/2$.

Therefore, for any time slot t, the total number of reporting users within any τ_{opt}-length window for PLPA satisfies

$$\sum_{\tau=t-\tau_{opt}+1}^{t} |U_\tau| = \sum_{\tau=t-\tau_{opt}+1}^{t} \left|U_{dis,\tau}\right| + \sum_{\tau=t-\tau_{opt}+1}^{t} \left|U_{pub,\tau}\right| \le n.$$

□

1) Proof of Theorem IV.2:

Proof. According to Lemma IX.4, for both PLPD and PLPA, each user participates at most once within any τ_{opt} consecutive time slots. Moreover, according to OPS, the effective privacy budget of user u_i is

$$\epsilon'_{opt,i} = \frac{\epsilon_i}{\lceil w_i/\tau_{opt} \rceil}.$$

Since each report is perturbed by GPRR under its corresponding effective privacy budget, PLPD and PLPA satisfy $\boldsymbol{\epsilon}'_{opt}$-PLDP within any τ_{opt}-length window.

According to Lemma IX.1, both mechanisms therefore satisfy $(\tau_{opt} \cdot \mathbf{1}, \boldsymbol{\epsilon}'_{opt})$-EPLDP.

For each user u_i, the effective privacy budget satisfies

$$\epsilon'_{opt,i} = \begin{cases} \epsilon_i, & \text{if } w_i \le \tau_{opt}, \\ \dfrac{\epsilon_i}{\lceil w_i/\tau_{opt} \rceil}, & \text{otherwise.} \end{cases}$$

Therefore, by Lemma IX.2 with $\widehat{w} = \tau_{opt}$, both PLPD and PLPA satisfy $(\boldsymbol{w}, \boldsymbol{\epsilon})$-EPLDP. □

2) Proof of Theorem V.3:

Proof. We first show that, for each privacy group PG_r, every user participates at most once within any τ_r^* consecutive time slots in both PLPD$^+$ and PLPA$^+$.

For privacy group PG_r, let $n_r = |U_r|$. According to GA-OPS with $\rho = 1/2$, we have $n_{sml,r} = \lfloor n_r/2 \rfloor$. Let $z^*_{opt,dis,r}$ and τ^*_r denote the optimal sampling size and recycling period returned by GA-OPS, respectively. By the definition of τ^*_r, we have

$$\tau^*_r z^*_{opt,dis,r} \le n_{sml,r} \le \left\lfloor \frac{n_r}{2} \right\rfloor.$$

Therefore, within any τ^*_r consecutive time slots, the total number of users assigned to the dissimilarity mechanism satisfies

$$\sum_{\tau=t-\tau^*_r+1}^{t} \left|U_{dis,r,\tau}\right| \le \left\lfloor \frac{n_r}{2} \right\rfloor.$$

For PLPD^+, the publication mechanism maintains a group-specific remaining publication population. At time slot t, the potential publication population satisfies $n_{pp,r} \le n_{pub,r}$, where

$$n_{pub,r} = \left\lfloor \frac{n_r}{2} \right\rfloor - \sum_{\tau=t-\tau^*_r+1}^{t-1} \left|U_{pub,r,\tau}\right|.$$

Hence, whether or not a new publication is generated, we have

$$\sum_{\tau=t-\tau^*_r+1}^{t} \left|U_{pub,r,\tau}\right| \le \left\lfloor \frac{n_r}{2} \right\rfloor.$$

Combining the dissimilarity and publication populations gives

$$\sum_{\tau=t-\tau^*_r+1}^{t} \left(\left|U_{dis,r,\tau}\right| + \left|U_{pub,r,\tau}\right|\right) \le 2\left\lfloor \frac{n_r}{2} \right\rfloor \le n_r.$$

For PLPA^+, the dissimilarity mechanism is identical to that of PLPD^+. We therefore only need to consider the publication mechanism. The population-absorption procedure is applied independently within each privacy group and has the same structure as that of PLPA, with the unified parameters replaced by the group-specific parameters n_r, $z^*_{opt,dis,r}$, and τ^*_r.

Fix any privacy group PG_r. Assume that there are c_r new publications of PG_r within a τ^*_r-length window, occurring at time slots $\tau_{r,1}, \tau_{r,2}, \ldots, \tau_{r,c_r}$. For the j-th publication, let $\eta_{r,j}$ denote the number of absorbed population shares. Thus, the j-th publication consumes $(1+\eta_{r,j})z^*_{opt,dis,r}$ users and nullifies $\eta_{r,j}$ subsequent population shares.

If $c_r = 1$, the publication consumes at most $\tau^*_r z^*_{opt,dis,r} \le \lfloor n_r/2 \rfloor$ users. For $c_r \ge 2$, the symmetry between the absorbed and nullified population shares gives

$$\tau^*_r \ge \sum_{j=1}^{c_r}(1+2\eta_{r,j}) - \eta_{r,1} - \eta_{r,c_r}.$$

Thus,

$$\begin{aligned}
\sum_{\tau=t-\tau^*_r+1}^{t} \left|U_{pub,r,\tau}\right| &\le z^*_{opt,dis,r} \sum_{j=1}^{c_r}(1+\eta_{r,j}) \\
&\le \frac{n_{sml,r}}{\tau^*_r} \sum_{j=1}^{c_r}(1+\eta_{r,j}) \\
&\le n_{sml,r} \frac{\sum_{j=1}^{c_r}(1+\eta_{r,j})}{\sum_{j=1}^{c_r}(1+2\eta_{r,j}) - \eta_{r,1} - \eta_{r,c_r}} \\
&= n_{sml,r} \frac{\sum_{j=1}^{c_r}(1+\eta_{r,j})}{\sum_{j=1}^{c_r}(1+\eta_{r,j}) + \sum_{j=2}^{c_r-1}\eta_{r,j}} \\
&\le n_{sml,r} \\
&\le \left\lfloor \frac{n_r}{2} \right\rfloor.
\end{aligned}$$

Therefore, the total number of dissimilarity and publication users of PG_r within any τ^*_r consecutive time slots is no more than n_r for both PLPD^+ and PLPA^+.

In both mechanisms, once a user is selected for reporting, the user is removed from the available-user set $U_{A,r}$ and is recycled only after the corresponding τ^*_r-length recycling period. Therefore, every user in U_r participates at most once within any τ^*_r consecutive time slots.

According to GA-OPS, for each privacy group $PG_r = (\widetilde{w}_r, \widetilde{\epsilon}_r, U_r)$, the effective privacy budget is

$$\epsilon'_r = \frac{\widetilde{\epsilon}_r}{\lceil \widetilde{w}_r/\tau^*_r \rceil}.$$

Each selected user in U_r applies GPRR with the effective privacy budget ϵ'_r, and thus each local report satisfies ϵ'_r-LDP.

Moreover, after the locally private reports are collected, all the remaining operations in PLPD^+ and PLPA^+, including the dissimilarity calculation, PAA-CW, PDEC, BA-PUE, population-weighted aggregation, simplex projection, and the reuse of historical publications, operate only on locally private outputs and therefore introduce no additional privacy loss.

Therefore, all the conditions in Lemma IX.3 are satisfied with $\tau_r = \tau^*_r$ for every PG_r. According to Lemma IX.3, both PLPD^+ and PLPA^+ satisfy $(\boldsymbol{w}, \boldsymbol{\epsilon})$-EPLDP. □

F. Proofs of Theorems for Utility Analysis

Lemma IX.5. *Given a privacy budget-count set $P = \{(\epsilon_k, n_k)\}_{k=1}^m$ satisfying $\sum_{k=1}^m n_k = n$, where ϵ_k is the k-th distinct privacy budget and n_k is the number of users with privacy budget ϵ_k. For a single privacy-budget group with population size N, privacy budget ϵ, and sampling size $z \le N$, the error upper bound of the FO process is*

$$\widetilde{err}_{FO}(N, z, \epsilon) = \frac{N-z}{z(N-1)} + \frac{d-1}{z} \cdot \frac{2e^\epsilon + d - 2}{(e^\epsilon - 1)^2}. \tag{25}$$

Let $\epsilon_{min} = \min_{k\in[m]} \epsilon_k$ denote the minimum privacy budget among all privacy-budget groups. The error upper bound of PFO *process is:*

$$\widetilde{err}_{PFO}(P, z) = \frac{n-z}{z(n-1)} + \frac{d-1}{z} \cdot \frac{2e^{\epsilon_{min}} + d - 2}{\left(e^{\epsilon_{min}} - 1\right)^2}. \tag{26}$$

Proof. For the FO case, all z sampled users are drawn from a single privacy-budget group with population size N. Hence, the GPRR perturbation variance reduces to $V_{LDP}(\epsilon, z, d) = \frac{d\lambda+\mu}{z}$. According to the sampling-variance upper bound in Equation (20) and the GPRR perturbation variance, we have

$$\begin{aligned}
err_{FO}(N, z, \epsilon) &\le \frac{N-z}{z(N-1)} + \frac{1}{z}(d\lambda + \mu) \\
&= \frac{N-z}{z(N-1)} + \frac{d-1}{z}\left(\frac{2}{e^\epsilon - 1} + \frac{d}{(e^\epsilon - 1)^2}\right) \\
&= \frac{N-z}{z(N-1)} + \frac{d-1}{z} \cdot \frac{2e^\epsilon + d - 2}{(e^\epsilon - 1)^2}.
\end{aligned}$$

For the PFO case, according to Equation (21), we have

$$err_{PFO}(P, z) = \frac{n-z}{z(n-1)} + \frac{1/z}{\sum_{k=1}^m 1/(d\lambda_k + \mu_k)}.$$

For each privacy-budget group k, according to the definitions of λ_k and μ_k, we have

$$d\lambda_k + \mu_k = \frac{d-1}{g_k}\left(\frac{2}{e^{\epsilon_k} - 1} + \frac{d}{(e^{\epsilon_k} - 1)^2}\right) = \frac{d-1}{g_k} \cdot \frac{2e^{\epsilon_k} + d - 2}{(e^{\epsilon_k} - 1)^2}.$$

Let $h(\epsilon) = \frac{2e^{\epsilon}+d-2}{(e^{\epsilon}-1)^2}$. Since $h(\epsilon)$ is monotonically decreasing for $\epsilon > 0$, for every $\epsilon_k \in \widetilde{\boldsymbol{\epsilon}}$, we have $h(\epsilon_k) \le h(\epsilon_{min})$. Therefore,

$$d\lambda_k + \mu_k \le \frac{d-1}{g_k} h(\epsilon_{min}),$$

and hence

$$\frac{1}{d\lambda_k + \mu_k} \ge \frac{g_k}{(d-1)h(\epsilon_{min})}.$$

Since $\sum_{k=1}^{m} g_k = 1$, we obtain

$$\sum_{k=1}^{m} \frac{1}{d\lambda_k + \mu_k} \ge \frac{1}{(d-1)h(\epsilon_{min})}.$$

Thus,

$$\begin{aligned} err_{PFO}(P, z) &\le \frac{n-z}{z(n-1)} + \frac{d-1}{z} h(\epsilon_{min}) \\ &= \frac{n-z}{z(n-1)} + \frac{d-1}{z} \cdot \frac{2e^{\epsilon}min + d - 2}{(e^{\epsilon}min - 1)^2}. \end{aligned}$$

□

Lemma IX.6. *Let $\widehat{\boldsymbol{f}}_{dis,t}$ be the PFO estimate used by the private dissimilarity mechanism at time slot t, and let $V_{dis,t} = \mathbb{E}\left[\left\|\widehat{\boldsymbol{f}}_{dis,t} - \boldsymbol{f}_t\right\|_2^2\right]$. Let $P_{opt} = \{(\epsilon_k, n_k)\}_{k=1}^{m}$ denote the effective privacy-budget count set after OPS, and let $\epsilon_{opt,min} = \min(\widetilde{\boldsymbol{\epsilon}}'_{\text{opt}})$. Conditioned on the publication history before time slot t, define $R_{opt} = \frac{e^{\epsilon}opt,min+d-2}{e^{\epsilon}opt,min-1}$. Then the variance of the dissimilarity satisfies $Var[dis - dis^*] \le \frac{9R_{opt}^2}{d} \widetilde{err}_{PFO}(P_{opt}, z_{opt})$, where $\widetilde{err}_{PFO}(P_{opt}, z_{opt}) = \frac{n-z_{opt}}{z_{opt}(n-1)} + \frac{d-1}{z_{opt}} \cdot \frac{2e^{\epsilon_{opt,min}}+d-2}{(e^{\epsilon_{opt,min}}-1)^2}$*

Proof. Let $\boldsymbol{e}_t = \widehat{\boldsymbol{f}}_{dis,t} - \boldsymbol{f}_t$, $\boldsymbol{\Delta}_t = \boldsymbol{f}_t - \widehat{\boldsymbol{f}}_l$. Conditioned on the previous publication history, $\widehat{\boldsymbol{f}}_l$ is fixed. Since $\widehat{\boldsymbol{f}}_{dis,t}$ is unbiased, $\mathbb{E}[\boldsymbol{e}_t] = \boldsymbol{0}$, and $V_{dis,t} = \mathbb{E}\left[\|\boldsymbol{e}_t\|_2^2\right]$.

According to the definitions of *dis* and *dis**, we have

$$\begin{aligned} dis - dis^* &= \frac{1}{d}\left\|\widehat{\boldsymbol{f}}_{dis,t} - \widehat{\boldsymbol{f}}_l\right\|_2^2 - \frac{1}{d}V_{dis,t} - \frac{1}{d}\|\boldsymbol{\Delta}_t\|_2^2 \\ &= \frac{1}{d}\left\|\widehat{\boldsymbol{f}}_{dis,t} - \boldsymbol{f}_t + \boldsymbol{f}_t - \widehat{\boldsymbol{f}}_l\right\|_2^2 - \frac{1}{d}V_{dis,t} - \frac{1}{d}\|\boldsymbol{\Delta}_t\|_2^2 \\ &= \frac{1}{d}\|\boldsymbol{e}_t + \boldsymbol{\Delta}_t\|_2^2 - \frac{1}{d}V_{dis,t} - \frac{1}{d}\|\boldsymbol{\Delta}_t\|_2^2 \\ &= \frac{2}{d}\boldsymbol{\Delta}_t^{\top}\boldsymbol{e}_t + \frac{1}{d}\left(\|\boldsymbol{e}_t\|_2^2 - V_{dis,t}\right). \end{aligned}$$

For GPRR with privacy budget ϵ_b, according to Equation (11), we have $\widehat{f}_{b,j} = \frac{h_{b,j}-q_b}{p_b-q_b}$ and $0 \le h_{b,j} \le 1$. Thus, $\widehat{f}_{b,j} \in \left[-\frac{1}{e^{\epsilon_b}-1}, \frac{e^{\epsilon_b}+d-2}{e^{\epsilon_b}-1}\right]$. Because PFO combines the group-level estimates using non-negative weights $\{\eta_b\}_{b=1}^{m}$ with $\sum_{b=1}^{m}\eta_b = 1$, and $\epsilon_b \ge \epsilon_{opt,min}$, we have $\widehat{\boldsymbol{f}}_{dis,t}[j] = \widehat{f}_j \in \left[-\frac{1}{e^{\epsilon}opt,min-1}, \frac{e^{\epsilon}opt,min+d-2}{e^{\epsilon}opt,min-1}\right]$. Since $0 \le \boldsymbol{f}_t[j] \le 1$, we further have

$$\begin{aligned} |e_t[j]| &= \left|\widehat{\boldsymbol{f}}_{dis,t}[j] - \boldsymbol{f}_t[j]\right| \\ &\le \max\left\{\left|-\frac{1}{e^{\epsilon_{opt,min}}-1} - 1\right|, \frac{e^{\epsilon_{opt,min}}+d-2}{e^{\epsilon_{opt,min}}-1}\right\} \\ &= \max\left\{\frac{e^{\epsilon_{opt,min}}}{e^{\epsilon_{opt,min}}-1}, \frac{e^{\epsilon_{opt,min}}+d-2}{e^{\epsilon_{opt,min}}-1}\right\} \\ &= R_{opt}. \end{aligned}$$

Similarly, $|\Delta_t[j]| \le R_{opt}$. Hence $\|\boldsymbol{e}_t\|_2^2 \le dR_{opt}^2$, $\|\boldsymbol{\Delta}_t\|_2^2 \le dR_{opt}^2$.

By the Cauchy–Schwarz inequality, we have

$$\mathbb{E}\left[(\boldsymbol{\Delta}_t^{\top}\boldsymbol{e}_t)^2\right] \le \|\boldsymbol{\Delta}_t\|_2^2 \mathbb{E}\|\boldsymbol{e}_t\|_2^2 \le dR_{opt}^2 V_{dis,t}.$$

Since $\text{Var}[X] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 \le \mathbb{E}\left[X^2\right]$, we have

$$\begin{aligned} \text{Var}\left[\|\boldsymbol{e}_t\|_2^2\right] &\le \mathbb{E}\left[\|\boldsymbol{e}_t\|_2^4\right] \\ &\le dR_{opt}^2 \mathbb{E}\left[\|\boldsymbol{e}_t\|_2^2\right] \\ &= dR_{opt}^2 V_{dis,t}. \end{aligned}$$

Therefore, by the Triangle Inequality and Equation (26) in Lemma IX.5, we have

$$\begin{aligned} \text{Var}\left[dis - dis^*\right] &= \mathbb{E}\left[(dis - dis^*)^2\right] \\ &\le \left(\frac{2}{d}\sqrt{\mathbb{E}\left[(\boldsymbol{\Delta}_t^{\top}\boldsymbol{e}_t)^2\right]} + \frac{1}{d}\sqrt{\text{Var}\left[\|\boldsymbol{e}_t\|_2^2\right]}\right)^2 \\ &\le \left(3R_{opt}\sqrt{\frac{V_{dis,t}}{d}}\right)^2 \\ &= \frac{9R_{opt}^2}{d}V_{dis,t} \\ &\le \frac{9R_{opt}^2}{d}\widetilde{err}_{PFO}(P_{opt}, z_{opt}) \end{aligned}$$

□

For the utility analysis of the group-aware private dissimilarity, we first introduce several auxiliary quantities. For each privacy group $PG_r = (\widetilde{w}_r, \widetilde{\epsilon}_r, U_r)$, let $n_{sml,r} = \lfloor n_r/2 \rfloor$, and define

$$\begin{aligned} \overline{V}_{dis,r} &= \widetilde{err}_{FO}\left(n_{sml,r}, z^*_{opt,dis,r}, \epsilon'_r\right), \\ \theta_{dis,r} &= \frac{n_{sml,r} - z^*_{opt,dis,r}}{z^*_{opt,dis,r}\left(n_{sml,r} - 1\right)}, \\ R_r &= \frac{e^{\epsilon'_r} + d - 2}{e^{\epsilon'_r} - 1}. \end{aligned}$$

Define the population-level variance upper bound and range coefficient as

$$\begin{aligned} \overline{V}_{dis} &= \sum_{r=1}^{K} \pi_r^2 \overline{V}_{dis,r}, \\ R_{grp} &= \sum_{r=1}^{K} \pi_r R_r. \end{aligned}$$

For each PG_r, further define

$$Q_r = \frac{\theta_{dis,r}}{1 - \theta_{dis,r}} R_r \sqrt{d\left(1 + \overline{V}_{dis,r}\right)}.$$

Finally, define the vector-level private-dissimilarity penalty as

$$D_{dis} = 3R_{grp}\sqrt{d\overline{V}_{dis}} + \sqrt{\sum_{r=1}^{K} \pi_r^4 Q_r^2}.$$

Lemma IX.7. *For the group-aware private dissimilarity used by* PLPD$^+$ *and* PLPA$^+$*, conditioned on the publication history before time slot t, we have*

$$d \cdot \mathbb{E}\left[\left|dis - dis^*\right|\right] \le D_{dis}.$$

Proof. For each privacy group PG_r, define

$$\begin{aligned} \boldsymbol{e}_{r,t} &= \widehat{\boldsymbol{f}}^{(r)}_{dis,t} - \boldsymbol{f}^{(r)}_t, \\ V_{dis,r,t} &= \mathbb{E}\left[\|\boldsymbol{e}_{r,t}\|_2^2\right]. \end{aligned}$$

Since the group-level FO estimate is unbiased, $\mathbb{E}[\boldsymbol{e}_{r,t}] = 0$. The population-weighted estimation error is

$$\begin{aligned}\boldsymbol{e}_t &= \widehat{\boldsymbol{f}}_{dis,t} - \boldsymbol{f}_t \\ &= \sum_{r=1}^{K} \pi_r \boldsymbol{e}_{r,t}.\end{aligned}$$

Since different privacy groups contain disjoint users and use independent local randomization, their estimation errors are independent. Hence,

$$\begin{aligned}V_{dis,t} &= \mathbb{E}\left[\|\boldsymbol{e}_t\|_2^2\right] \\ &= \sum_{r=1}^{K} \pi_r^2 V_{dis,r,t} \\ &\le \sum_{r=1}^{K} \pi_r^2 \overline{V}_{dis,r} = \overline{V}_{dis},\end{aligned}$$

where the inequality follows from Lemma IX.5.

For the GPRR estimate of PG_r, each coordinate satisfies

$$\widehat{f}^{(r)}_{dis,t}[j] \in \left[-\frac{1}{e^{\epsilon'_r}-1}, \frac{e^{\epsilon'_r}+d-2}{e^{\epsilon'_r}-1}\right].$$

Therefore, $|e_{r,t}[j]| \le R_r$, and thus

$$|e_t[j]| \le \sum_{r=1}^{K} \pi_r R_r = R_{grp}.$$

Let $\boldsymbol{\Delta}_t = \boldsymbol{f}_t - \widehat{\boldsymbol{f}}_l$. Since the released estimate is projected onto the probability simplex, both $\boldsymbol{f}_t$ and $\widehat{\boldsymbol{f}}_l$ are probability distributions. Hence, $|\Delta_t[j]| \le 1 \le R_{grp}$. Let

$$\widehat{V}_{dis,t} = \sum_{r=1}^{K} \pi_r^2 \widehat{V}_{dis,r,t}.$$

By Theorem V.1, $\mathbb{E}\left[\widehat{V}_{dis,r,t}\right] = V_{dis,r,t}$, and therefore $\mathbb{E}\left[\widehat{V}_{dis,t}\right] = V_{dis,t}$.

According to the definitions of *dis* and *dis**,

$$\begin{aligned}dis - dis^* = \frac{2}{d}\boldsymbol{\Delta}_t^\top \boldsymbol{e}_t + \frac{1}{d}\left(\|\boldsymbol{e}_t\|_2^2 - V_{dis,t}\right) \\ - \frac{1}{d}\left(\widehat{V}_{dis,t} - V_{dis,t}\right).\end{aligned}$$

Moreover, according to the above decomposition, $\mathbb{E}[\boldsymbol{e}_t] = 0$, $\mathbb{E}\left[\|\boldsymbol{e}_t\|_2^2\right] = V_{dis,t}$, and $\mathbb{E}\left[\widehat{V}_{dis,t}\right] = V_{dis,t}$. Therefore,

$$\mathbb{E}\left[dis - dis^*\right] = 0.$$

For the first term, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\mathbb{E}\left[\left(\boldsymbol{\Delta}_t^\top \boldsymbol{e}_t\right)^2\right] &\le \|\boldsymbol{\Delta}_t\|_2^2 \, \mathbb{E}\left[\|\boldsymbol{e}_t\|_2^2\right] \\ &\le dR_{grp}^2 V_{dis,t}.\end{aligned}$$

Moreover, $\|\boldsymbol{e}_t\|_2^2 \le dR_{grp}^2$, and thus

$$\begin{aligned}\mathrm{Var}\left[\|\boldsymbol{e}_t\|_2^2\right] &\le \mathbb{E}\left[\|\boldsymbol{e}_t\|_2^4\right] \\ &\le dR_{grp}^2 V_{dis,t}.\end{aligned}$$

It remains to bound the uncertainty introduced by the unbiased variance estimator. According to Theorem V.1,

$$\widehat{V}_{dis,r,t} = \frac{\theta_{dis,r}\left(1 - \left\|\widehat{\boldsymbol{f}}^{(r)}_{dis,t}\right\|_2^2\right) + V_{LDP}\left(\epsilon'_r, z^*_{opt,dis,r}, d\right)}{1 - \theta_{dis,r}}.$$

Therefore,

$$\mathrm{Var}\left[\widehat{V}_{dis,r,t}\right] = \left(\frac{\theta_{dis,r}}{1-\theta_{dis,r}}\right)^2 \mathrm{Var}\left[\left\|\widehat{\boldsymbol{f}}^{(r)}_{dis,t}\right\|_2^2\right].$$

Since $\left|\widehat{f}^{(r)}_{dis,t}[j]\right| \le R_r$, we have

$$\begin{aligned}\mathrm{Var}\left[\left\|\widehat{\boldsymbol{f}}^{(r)}_{dis,t}\right\|_2^2\right] &\le \mathbb{E}\left[\left\|\widehat{\boldsymbol{f}}^{(r)}_{dis,t}\right\|_2^4\right] \\ &\le dR_r^2 \mathbb{E}\left[\left\|\widehat{\boldsymbol{f}}^{(r)}_{dis,t}\right\|_2^2\right] \\ &= dR_r^2\left(\left\|\boldsymbol{f}_t^{(r)}\right\|_2^2 + V_{dis,r,t}\right) \\ &\le dR_r^2\left(1 + \overline{V}_{dis,r}\right).\end{aligned}$$

Hence,

$$\mathrm{Var}\left[\widehat{V}_{dis,r,t}\right] \le Q_r^2. \tag{27}$$

Since the group-level estimators are independent,

$$\begin{aligned}\mathrm{Var}\left[\widehat{V}_{dis,t}\right] &= \sum_{r=1}^{K} \pi_r^4 \, \mathrm{Var}\left[\widehat{V}_{dis,r,t}\right] \\ &\le \sum_{r=1}^{K} \pi_r^4 Q_r^2.\end{aligned}$$

Therefore, by the triangle inequality in L_2,

$$\begin{aligned}\sqrt{\mathrm{Var}\left[dis - dis^*\right]} &\le 3R_{grp}\sqrt{\frac{V_{dis,t}}{d}} + \frac{1}{d}\sqrt{\sum_{r=1}^{K}\pi_r^4 Q_r^2} \\ &\le 3R_{grp}\sqrt{\frac{\overline{V}_{dis}}{d}} + \frac{1}{d}\sqrt{\sum_{r=1}^{K}\pi_r^4 Q_r^2}.\end{aligned}$$

Consequently,

$$\begin{aligned}d \cdot \mathbb{E}\left[\left|dis - dis^*\right|\right] &\le d\sqrt{\mathrm{Var}\left[dis - dis^*\right]} \\ &\le D_{dis}.\end{aligned}$$

□

Lemma IX.8. *For each privacy group PG_r, let $n_{sml,r} = \lfloor n_r/2 \rfloor$. Under the group-wise uniform-publication assumption of Theorem V.4, the average vector-level publication-variance upper bound of PG_r satisfies*

$$\begin{aligned}\overline{V}_{pub,r} = \frac{2\left(2^{s_r}-1\right)}{s_r}\left(\frac{1}{n_{sml,r}-1} + \frac{d-1}{n_{sml,r}} \cdot \frac{2e^{\epsilon'_r}+d-2}{\left(e^{\epsilon'_r}-1\right)^2}\right) \\ - \frac{1}{n_{sml,r}-1}.\end{aligned}$$

Proof. At the beginning of a recycling cycle, PLPD$^+$ reserves $n_{sml,r} = \lfloor n_r/2 \rfloor$ users of PG_r for the publication mechanism. According to Algorithm 5, each new publication uses half of the remaining publication population. Ignoring integer rounding, the j-th publication therefore uses

$$z_{r,j} = \frac{n_{sml,r}}{2^j}, \qquad j \in [s_r].$$

According to the FO error upper bound from Equation (25) in Lemma IX.5 the vector-level variance of the j-th group-level publication estimate is upper-bounded by

$$\widetilde{err}_{FO}\left(n_{sml,r}, z_{r,j}, \epsilon'_r\right) = \frac{n_{sml,r} - z_{r,j}}{z_{r,j}\left(n_{sml,r}-1\right)} + \frac{d-1}{z_{r,j}} \cdot \frac{2e^{\epsilon'_r}+d-2}{\left(e^{\epsilon'_r}-1\right)^2}.$$

Substituting $z_{r,j} = n_{sml,r}/2^j$ yields

$$\widetilde{err}_{FO}\left(n_{sml,r}, \frac{n_{sml,r}}{2^j}, \epsilon_r'\right) = \frac{2^j - 1}{n_{sml,r} - 1} + \frac{2^j(d-1)}{n_{sml,r}} \cdot \frac{2e^{\epsilon_r'} + d - 2}{\left(e^{\epsilon_r'} - 1\right)^2}.$$

Under the uniform-publication assumption, the s_r publication stages are equally represented in the time average. Hence,

$$\overline{V}_{pub,r} = \frac{1}{s_r} \sum_{j=1}^{s_r} \widetilde{err}_{FO}\left(n_{sml,r}, \frac{n_{sml,r}}{2^j}, \epsilon_r'\right).$$

Since $\sum_{j=1}^{s_r} 2^j = 2(2^{s_r} - 1)$, we obtain

$$\overline{V}_{pub,r} = \frac{2\left(2^{s_r} - 1\right)}{s_r}\left(\frac{1}{n_{sml,r} - 1} + \frac{d-1}{n_{sml,r}} \cdot \frac{2e^{\epsilon_r'} + d - 2}{\left(e^{\epsilon_r'} - 1\right)^2}\right) - \frac{1}{n_{sml,r} - 1}.$$

□

Lemma IX.9 (Shift-Bounded Fusion Bias)**.** *Let* $\Xi_t \triangleq \max_{r,s\in[K]} \|\boldsymbol{f}_t^{(r)} - \boldsymbol{f}_t^{(s)}\|_2 \in [0, \sqrt{2}]$. *For any* $\boldsymbol{\gamma}, \boldsymbol{\pi}$ *in the probability simplex,*

$$\left\|\sum_{r=1}^{K} (\gamma_r - \pi_r)\boldsymbol{f}_t^{(r)}\right\|_2 \le \tfrac{1}{2}\|\boldsymbol{\gamma} - \boldsymbol{\pi}\|_1 \, \Xi_t \le \Xi_t.$$

Proof. Let $a_r = \gamma_r - \pi_r$. Since $\sum_r a_r = 0$, let $P = \sum_{a_r>0} a_r = \frac{1}{2}\|\boldsymbol{a}\|_1$; the case $P = 0$ is trivial. Then $\sum_r a_r \boldsymbol{f}_t^{(r)} = P(\boldsymbol{x} - \boldsymbol{y})$, where $\boldsymbol{x} = \sum_{a_r>0} \frac{a_r}{P} \boldsymbol{f}_t^{(r)}$ and $\boldsymbol{y} = \sum_{a_s<0} \frac{|a_s|}{P} \boldsymbol{f}_t^{(s)}$ are convex combinations. Hence $\boldsymbol{x} - \boldsymbol{y} = \sum_{r,s} \alpha_r \beta_s (\boldsymbol{f}_t^{(r)} - \boldsymbol{f}_t^{(s)})$ with $\sum_{r,s} \alpha_r \beta_s = 1$, and the triangle inequality gives $\|\boldsymbol{x} - \boldsymbol{y}\|_2 \le \Xi_t$. Finally $P \le 1$. □

1) Proof of Theorems IV.3 and IV.4:

Proof. We first assume the $\mathcal{M}_{dis,t}$ is not private for both PLPD and PLPA and get the error of $\mathcal{M}_{pub,t}$. After that, we analyze the error of $\mathcal{M}_{dis,t}$ when it is private.

Define $A_{opt} = \frac{d-1}{n} \frac{2e^{\epsilon_{opt,min}} + d - 2}{(e^{\epsilon_{opt,min}} - 1)^2}$. According to Equation (26) in Lemma IX.5, for any sampling size z, we have

$$\begin{aligned} err_{PFO}(P_{opt}, z) &\le \widetilde{err}_{PFO}(P_{opt}, z) \\ &= \frac{n - z}{z(n-1)} + \frac{d-1}{z} \cdot \frac{2e^{\epsilon_{opt,min}} + d - 2}{(e^{\epsilon_{opt,min}} - 1)^2} \\ &= \frac{n}{z}\left(\frac{1}{n-1} + A_{opt}\right) - \frac{1}{n-1}. \end{aligned}$$

We first analyze the publication mechanism $\mathcal{M}_{pub,t}$ in PLPD. Ignoring integer rounding, the population sizes of the successive new publications are $\frac{n}{2^2}, \frac{n}{2^3}, \ldots, \frac{n}{2^{s_{opt}+1}}$. Under the uniform-publication assumption, each publication represents τ_{opt}/s_{opt} time slots. Therefore, the average publication error satisfies

$$\begin{aligned} \overline{err}_{\mathrm{PLPD},pub} &\le \frac{1}{s_{opt}} \sum_{j=1}^{s_{opt}} \widetilde{err}_{PFO}\left(P_{opt}, \frac{n}{2^{j+1}}\right) \\ &= \frac{1}{s_{opt}} \sum_{j=1}^{s_{opt}} \left[2^{j+1}\left(\frac{1}{n-1} + A_{opt}\right) - \frac{1}{n-1}\right] \\ &= \frac{4(2^{s_{opt}} - 1)}{s_{opt}}\left(\frac{1}{n-1} + A_{opt}\right) - \frac{1}{n-1}. \end{aligned}$$

For the publication mechanism $\mathcal{M}_{pub,t}$ in PLPA, we further assume that each new publication is associated with the same number of skipped and nullified time slots. Let $B = \frac{\tau_{opt} - s_{opt}}{2s_{opt}}$ denote the number of skipped (and equivalently, nullified) time slots associated with each new publication, and let $C = \frac{s_{opt}}{\tau_{opt}} = \frac{1}{2B+1}$. Thus, under the uniform-publication assumption, each publication cycle consists of B skipped time slots, one new publication, and B nullified time slots, i.e., $2B + 1$ time slots in total. For analytical simplicity, we assume $B \in \mathbb{N}^+$. Consider one publication cycle. According to Algorithm 2, after $j - 1$ consecutive skipped time slots, the potential publication population at the j-th absorbable time slot is jz_{opt}. Since $\mathcal{M}_{dis,t}$ is temporarily assumed to be non-private, the dissimilarity used for the publication decision equals the true approximation error, i.e., $dis = dis^*$. According to Algorithm 2, a publication is skipped only when $dis \le err$. Therefore, the approximation error of the j-th skipped time slot is no larger than its potential reporting error, and is bounded by

$$\widetilde{err}_{PFO}\left(P_{opt}, j \cdot z_{opt}\right), \qquad j \in [B].$$

After B skipped time slots, the next time slot generates a new publication using $(B+1)z_{opt}$ users, whose error is bounded by $\widetilde{err}_{PFO}\left(P_{opt}, (B+1)z_{opt}\right)$. Moreover, the B nullified time slots associated with this publication contribute $B \cdot err_{nlf}$. Hence, the average publication error of PLPA satisfies

$$\begin{aligned} err_{\mathrm{PLPA},pub} &\le \frac{s_{opt}}{\tau_{opt}}\left[\sum_{j=1}^{B+1} \widetilde{err}_{PFO}\left(P_{opt}, jz_{opt}\right) + B \cdot err_{nlf}\right] \\ &= C\left[\sum_{j=1}^{B+1} \widetilde{err}_{PFO}\left(P_{opt}, jz_{opt}\right) + B \cdot err_{nlf}\right]. \end{aligned}$$

Using

$$\widetilde{err}_{PFO}(P_{opt}, z) = \frac{n}{z}\left(\frac{1}{n-1} + A_{opt}\right) - \frac{1}{n-1},$$

we further obtain

$$\begin{aligned} err_{PLPA,pub} &\le C \sum_{j=1}^{B+1}\left[\frac{n}{jz_{opt}}\left(\frac{1}{n-1} + A_{opt}\right) - \frac{1}{n-1}\right] + BC \cdot err_{nlf} \\ &= \frac{Cn}{z_{opt}}\left(\frac{1}{n-1} + A_{opt}\right) H_{B+1} - \frac{C(B+1)}{n-1} + BC \cdot err_{nlf}, \end{aligned}$$

where $H_{B+1} = \sum_{j=1}^{B+1} \frac{1}{j}$ is the $(B+1)$-th harmonic number.

When $\mathcal{M}_{dis,t}$ is private, the publication decision based on *dis* may differ from that based on the non-private dissimilarity dis^*. If the two decisions are the same, no additional error is introduced. Otherwise, there are two possible choosing error cases: (1) $dis^* \le err < dis$; and (2) $dis \le err < dis^*$. We refer to these two cases as *choosing errors*, and let $\mathbf{1}_{\mathrm{ce}} = 1$ when either of the two cases happens; and $\mathbf{1}_{\mathrm{ce}} = 0$ otherwise. When a choosing error occurs, the additional per-domain error-bound term is $\max\{dis^*, err\} - \min\{dis^*, err\} = |err - dis^*|$.

For the choosing error in both cases above, we have

$$|err - dis^*| \le |dis - dis^*|,$$

which means, in either case, the choosing error is at most $|dis - dis^*|$. Let $err_{dis,t}^{(per)}$ denote the expected additional per-domain error-bound term caused by a choosing error. Since

the utility bounds above are expressed in vector-level total error, we have

$$\begin{aligned} err_{dis,t} &= d \cdot err^{(per)}_{dis,t} \\ &= d \cdot \mathbb{E}\left[|err - dis^*| \cdot \mathbf{1}_{\mathrm{ce}}\right] \\ &\le d\,\mathbb{E}\left[|dis - dis^*|\right] \\ &\le d\sqrt{\mathbb{E}\left[(dis - dis^*)^2\right]} \\ &= d\sqrt{\mathrm{Var}[dis - dis^*]}. \end{aligned}$$

According to Lemma IX.6, we further have

$$\begin{aligned} err_{dis,t} &\le d\sqrt{\frac{9R_{opt}^2}{d}\widetilde{err}_{PFO}(P_{opt}, z_{opt})} \\ &= 3R_{opt}\sqrt{d\,\widetilde{err}_{PFO}(P_{opt}, z_{opt})} \\ &= 3R_{opt}\sqrt{\frac{nd}{z_{opt}}\left(\frac{1}{n-1} + A_{opt}\right) - \frac{d}{n-1}}. \end{aligned}$$

Therefore, the error upper bound is

$$\begin{aligned} &3R_{opt}\sqrt{\frac{nd}{z_{opt}}\left(\frac{1}{n-1} + A_{opt}\right) - \frac{d}{n-1}} \\ &+ \frac{4(2^{s}opt - 1)}{s_{opt}}\left(\frac{1}{n-1} + A_{opt}\right) - \frac{1}{n-1} \end{aligned}$$

in PLPD, and

$$\begin{aligned} &3R_{opt}\sqrt{\frac{nd}{z_{opt}}\left(\frac{1}{n-1} + A_{opt}\right) - \frac{d}{n-1}} + \frac{Cn}{z_{opt}}\left(\frac{1}{n-1} + A_{opt}\right) H_{B+1} \\ &- \frac{C(B+1)}{n-1} + BC \cdot err_{nlf} \end{aligned}$$

in PLPA. □

2) Proof of Theorem V.4:

Proof. Fix a time slot t satisfying the conditions of Theorem V.4, and treat the reuse configuration $(\boldsymbol{PG}_t^{(A)}, \boldsymbol{PG}_t^{(R)}, \{l_r\})$ as given. Write $\mathcal{A} = \boldsymbol{PG}_t^{(A)}$, $\mathcal{R} = \boldsymbol{PG}_t^{(R)}$, $\gamma_r = \gamma_{r,t}^{\mathrm{PLPA}^+}$, and let $\mathcal{F}_t^{\mathrm{Pre}}$ be the information available before the active-group publication reports are generated. By Lines 1–18 of Algorithm 6, $\{\gamma_r\}$, $\{\widehat{\boldsymbol{f}}_{pub,l_r}^{(r)}\}_{r\in\mathcal{R}}$, $\{\widehat{\boldsymbol{f}}_{dis,t}^{(r)}, \widehat{V}_{dis,r,t}\}_{r\in[K]}$, dis_t, and $\widetilde{err}_t^{+,PA}$ are all $\mathcal{F}_t^{\mathrm{Pre}}$-measurable.

Notation. Define $\boldsymbol{e}_r = \widehat{\boldsymbol{f}}_{dis,t}^{(r)} - \boldsymbol{f}_t^{(r)}$ for $r \in [K]$, $\boldsymbol{u}_r = \widehat{\boldsymbol{f}}_{pub,l_r}^{(r)} - \boldsymbol{f}_t^{(r)}$ for $r \in \mathcal{R}$, and $c_r = \gamma_r - \pi_r$ if $r \in \mathcal{A}$, $c_r = -\pi_r$ if $r \in \mathcal{R}$. Let

$$\boldsymbol{B}_t = \sum_{r\in\mathcal{A}}(\gamma_r - \pi_r)\boldsymbol{f}_t^{(r)} + \sum_{r\in\mathcal{R}}(\gamma_r \widehat{\boldsymbol{f}}_{pub,l_r}^{(r)} - \pi_r \boldsymbol{f}_t^{(r)}), \qquad \boldsymbol{E}_t = \sum_{r=1}^{K} c_r \boldsymbol{e}_r.$$

Since $\sum_r \gamma_r = \sum_r \pi_r = 1$,

$$\boldsymbol{B}_t = \sum_{r=1}^{K}(\gamma_r - \pi_r)\boldsymbol{f}_t^{(r)} + \sum_{r\in\mathcal{R}}\gamma_r \boldsymbol{u}_r, \tag{28}$$

and, with $\boldsymbol{h}_t^{(r)} = \widehat{\boldsymbol{f}}_{dis,t}^{(r)}$, the plug-in bias vector in

$$\begin{aligned} \widehat{B}_t^2 = \max\Big\{0, \Big\|B\left(\boldsymbol{H}_t, \widehat{\boldsymbol{F}}_{l,t}\right)\Big\|_2^2 - \sum_{r\in\boldsymbol{PG}_t^{(A)}}(\gamma_{r,t} - \pi_r)^2 \,\mathrm{Var}\left[\boldsymbol{h}_t^{(r)}\right] \\ - \sum_{r\in\boldsymbol{PG}_t^{(R)}}\left(\pi_r^2 \,\mathrm{Var}\left[\boldsymbol{h}_t^{(r)}\right] + \gamma_{r,t}^2 \,\mathrm{Var}\left[\widehat{\boldsymbol{f}}_{l_r}^{(r)}\right]\right)\Big\} \end{aligned}$$

satisfies

$$\boldsymbol{B}(\boldsymbol{H}_t, \widehat{\boldsymbol{F}}_{l,t}) = \boldsymbol{B}_t + \boldsymbol{E}_t.$$

Step 1 (conditional publication risk). Since $\boldsymbol{f}_t = \sum_r \pi_r \boldsymbol{f}_t^{(r)}$ and $\sum_r \gamma_r = 1$, the population-level estimate obtained by Equation (2) and Line 32 of Algorithm 5 satisfies

$$\widetilde{\boldsymbol{f}}_{pub,t} - \boldsymbol{f}_t = \sum_{r\in\mathcal{A}}\gamma_r(\widehat{\boldsymbol{f}}_{pub,t}^{(r)} - \boldsymbol{f}_t^{(r)}) + \boldsymbol{B}_t.$$

Given $\mathcal{F}_t^{\mathrm{Pre}}$, $\boldsymbol{B}_t$ is fixed, and the active-group publication errors are zero-mean and mutually independent, with variances upper-bounded by $V_{pub,r,t}^{\mathrm{Pre}}$ according to Equations (19) and (20). Hence

$$\mathbb{E}[\|\widetilde{\boldsymbol{f}}_{pub,t} - \boldsymbol{f}_t\|_2^2 \mid \mathcal{F}_t^{\mathrm{Pre}}] \le \sum_{r\in\mathcal{A}}\gamma_r^2 V_{pub,r,t}^{\mathrm{Pre}} + \|\boldsymbol{B}_t\|_2^2 \triangleq \mathcal{R}_{pub,t}. \tag{29}$$

Since the Euclidean projection onto the probability simplex (Line 32 of Algorithm 5) is non-expansive, the released publication satisfies the same bound.

Step 2 (bounding $\mathbb{E}[\mathcal{R}_{\mathrm{pub},t}]$). Since $0 \le \gamma_r \le 1$, the first term of $\mathcal{R}_{pub,t}$ is at most $\sum_{r\in\mathcal{A}} V_{pub,r,t}^{\mathrm{Pre}}$. For $\boldsymbol{B}_t$, the first term of Equation (28) is bounded by Ξ according to Lemma IX.9. For the second term, since $\gamma_r \in [0,1]$ and $\sum_{r\in\mathcal{R}}\gamma_r \le 1$, the Cauchy–Schwarz inequality gives

$$\Big\|\sum_{r\in\mathcal{R}}\gamma_r \boldsymbol{u}_r\Big\|_2^2 \le \Big(\sum_{r\in\mathcal{R}}\gamma_r\Big)\Big(\sum_{r\in\mathcal{R}}\gamma_r\|\boldsymbol{u}_r\|_2^2\Big) \le \sum_{r\in\mathcal{R}}\|\boldsymbol{u}_r\|_2^2 \triangleq U_t^2.$$

Hence $\|\boldsymbol{B}_t\|_2 \le \Xi + U_t$ holds pointwise. Since the historical estimate of each reused group is unbiased for $\boldsymbol{f}_{l_r}^{(r)}$, $\mathbb{E}\|\boldsymbol{u}_r\|_2^2 = V_{pub,r,l_r} + \|\boldsymbol{f}_{l_r}^{(r)} - \boldsymbol{f}_t^{(r)}\|_2^2$, and thus $\mathbb{E}[U_t^2] = V_{hist,t} + \Gamma_t^2$. By the Minkowski inequality in L^2,

$$\sqrt{\mathbb{E}\|\boldsymbol{B}_t\|_2^2} \le \Xi + \sqrt{\mathbb{E}[U_t^2]} = \rho_t, \qquad \mathbb{E}[\mathcal{R}_{pub,t}] \le \sum_{r\in\mathcal{A}} V_{pub,r,t}^{\mathrm{Pre}} + \rho_t^2. \tag{30}$$

Step 3 (error of the selected action). Under the population lower-bound condition, the decision in Line 19 of Algorithm 6 depends only on the comparison between dis_t and $\widetilde{err}_t^{+,PA}$. If a new publication is triggered, its conditional error is at most $\mathcal{R}_{pub,t}$ by Step 1. Otherwise, $dis_t \le \widetilde{err}_t^{+,PA}$, and the reuse error satisfies

$$\begin{aligned} d \cdot dis_t^* &\le d \cdot dis_t + d|dis_t - dis_t^*| \\ &\le \mathcal{R}_{pub,t} + |d \cdot \widetilde{err}_t^{+,PA} - \mathcal{R}_{pub,t}| + d|dis_t - dis_t^*|. \end{aligned}$$

Thus, in both cases,

$$\begin{aligned} &\mathbb{E}[\|\widehat{\boldsymbol{f}}_{pub,t} - \boldsymbol{f}_t\|_2^2 \mid \mathcal{F}_t^{\mathrm{Pre}}] \\ \le &\mathcal{R}_{pub,t} + |d \cdot \widetilde{err}_t^{+,PA} - \mathcal{R}_{pub,t}| + d|dis_t - dis_t^*|. \end{aligned} \tag{31}$$

Step 4 (bounding the risk-estimation error). As instantiated in Line 18 of Algorithm 6, Equation (9) uses $V_{pub,r,t}^{\mathrm{Pre}}$ for active groups, $\widehat{V}_{dis,r,t}$ for $\mathrm{Var}[\boldsymbol{h}_t^{(r)}]$, and $\widehat{V}_{pub,r,l_r}$ for $\mathrm{Var}[\widehat{\boldsymbol{f}}_{l_r}^{(r)}]$. Let

$$X_t = \|\boldsymbol{B}_t + \boldsymbol{E}_t\|_2^2 - \sum_{r=1}^{K} c_r^2 \widehat{V}_{dis,r,t}, \qquad Y_t = \sum_{r\in\mathcal{R}}\gamma_r^2 \widehat{V}_{pub,r,l_r} \ge 0.$$

Then $\widehat{B}_t^2 = \max\{0, X_t - Y_t\}$, and therefore $d \cdot \widetilde{err}_t^{+,PA} = \sum_{r\in\mathcal{A}}\gamma_r^2 V_{pub,r,t}^{\mathrm{Pre}} + \max\{X_t, Y_t\}$. Comparing with Equation (29), and using the elementary inequality $|\max\{X,Y\} - T| \le |X - T| + Y$ for $T, Y \ge 0$,[1] we obtain

$$|d \cdot \widetilde{err}_t^{+,PA} - \mathcal{R}_{pub,t}| \le |X_t - \|\boldsymbol{B}_t\|_2^2| + Y_t,$$

[1] If $X \ge Y$, the left side equals $|X - T|$. If $X < Y$, then either $Y \ge T$, giving $Y - T \le Y$, or $Y < T$, giving $T - Y < T - X = |X - T|$.

where

$$X_t - \|\boldsymbol{B}_t\|_2^2 = 2\boldsymbol{B}_t^\top \boldsymbol{E}_t + \left(\|\boldsymbol{E}_t\|_2^2 - \sum_r c_r^2 V_{dis,r,t}\right) - \sum_r c_r^2 (\widehat{V}_{dis,r,t} - V_{dis,r,t}).$$

Let $\widetilde{\gamma}_r = \gamma_r \mathbf{1}_{r\in\mathcal{A}}$. Since $\widetilde{\boldsymbol{\gamma}}$ and $\boldsymbol{\pi}$ are non-negative with ℓ_1-norm at most one, $\|\boldsymbol{c}\|_2^2 \le \|\widetilde{\boldsymbol{\gamma}}\|_2^2 + \|\boldsymbol{\pi}\|_2^2 \le 2$ and $|c_r| \le 1$. By the Cauchy–Schwarz inequality, $\|\boldsymbol{E}_t\|_2^2 \le \|\boldsymbol{c}\|_2^2 \sum_r \|\boldsymbol{e}_r\|_2^2$, and hence $\mathbb{E}\|\boldsymbol{E}_t\|_2^2 \le 2S_{dis}$. We bound the four terms separately: (i) by the Cauchy–Schwarz inequality and Equation (30), $2\mathbb{E}|\boldsymbol{B}_t^\top \boldsymbol{E}_t| \le 2\sqrt{\mathbb{E}\|\boldsymbol{B}_t\|_2^2}\sqrt{\mathbb{E}\|\boldsymbol{E}_t\|_2^2} \le 2\sqrt{2}\,\rho_t\sqrt{S_{dis}}$; (ii) $\mathbb{E}\left|\|\boldsymbol{E}_t\|_2^2 - \sum_r c_r^2 V_{dis,r,t}\right| \le 2S_{dis} + S_{dis} = 3S_{dis}$; (iii) by Theorem V.1 and Equation (27), $\mathbb{E}\left|\sum_r c_r^2(\widehat{V}_{dis,r,t} - V_{dis,r,t})\right| \le \sum_r \sqrt{\mathrm{Var}[\widehat{V}_{dis,r,t}]} \le \sum_r Q_r$; (iv) by Theorem V.1, $\mathbb{E}[Y_t] \le \sum_{r\in\mathcal{R}} \mathbb{E}[\widehat{V}_{pub,r,l_r}] = V_{hist,t}$. Therefore,

$$\mathbb{E}|d\cdot \widetilde{err}_t^{+,PA} - \mathcal{R}_{pub,t}| \le 2\sqrt{2}\,\rho_t\sqrt{S_{dis}} + 3S_{dis} + \sum_{r=1}^{K} Q_r + V_{hist,t}. \tag{32}$$

Step 5 (combination). Taking total expectation of Equation (31), and applying Equations (30) and (32) together with Lemma VIII.7, i.e., $d\cdot\mathbb{E}|dis_t - dis_t^*| \le D_{dis}$, yields

$$err_t^{\mathrm{PLPA}^+,\mathrm{Pre}} \le \sum_{r\in\mathcal{A}} V_{pub,r,t}^{\mathrm{Pre}} + \rho_t^2 + 2\sqrt{2}\,\rho_t\sqrt{S_{dis}} + 3S_{dis} + \sum_{r=1}^{K} Q_r + V_{hist,t} + D_{dis},$$

which is the claimed result.

PLPD$^+$ *case.* In PLPD$^+$, all privacy groups are updated synchronously, i.e., $\mathcal{A} = [K]$ and $\mathcal{R} = \emptyset$. Repeating Steps 1–5 with Algorithm 5 and Equation (10) in place of Algorithm 6 and Equation (9), we have $\Gamma_t = V_{hist,t} = 0$ and $\rho_t = \Xi$, and thus

$$err_t^{\mathrm{PLPD}^+,\mathrm{Pre}} \le \sum_{r=1}^{K} V_{pub,r,t}^{\mathrm{Pre}} + \Xi^2 + 2\sqrt{2}\,\Xi\sqrt{S_{dis}} + D.$$

Averaging over time and applying Lemma VIII.8 to replace $V_{pub,r,t}^{\mathrm{Pre}}$ by $\overline{V}_{pub,r}$ yields the PLPD$^+$ bound in Theorem V.4. □

G. Additional Experiments

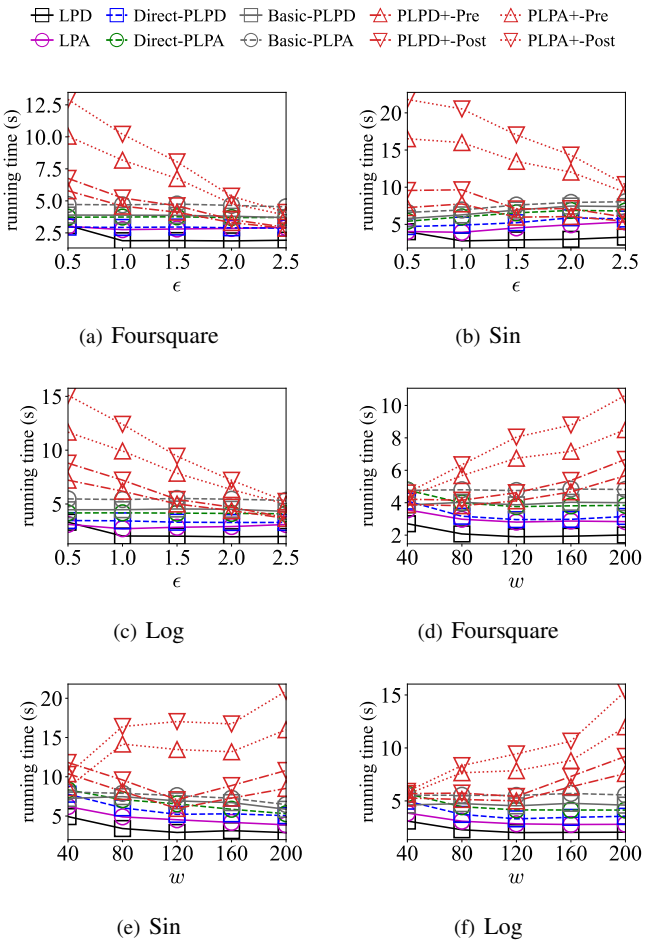
1) Running Time: Figure 8 reports the running time on Foursquare, Sin, and Log, which follows the same trends as Figure 4.

2) AJSD Results: We further evaluate each method in terms of the *Average Jensen-Shannon Divergence* [36] (*AJSD*, $\bar{D}_{JS}$), defined as:

$$\begin{aligned}\bar{D}_{JS}\left(\widehat{\boldsymbol{f}}\|\boldsymbol{f}\right) &= \frac{1}{T}\sum_{\tau=1}^{T} D_{JS}\left(\widehat{\boldsymbol{f}}\|\boldsymbol{f}\right)\\ &= \frac{1}{T}\sum_{\tau=1}^{T}\left(\frac{1}{2}D_{KL}\left(\widehat{\boldsymbol{f}}\|\boldsymbol{v}\right) + \frac{1}{2}D_{KL}\left(\boldsymbol{f}\|\boldsymbol{v}\right)\right)\\ &= \frac{1}{2T}\sum_{\tau=1}^{T}\sum_{j=1}^{d}\left(\widehat{\boldsymbol{f}}_{j,\tau}\log\left(\frac{\widehat{\boldsymbol{f}}_{j,\tau}}{\boldsymbol{v}_{j,\tau}}\right) + \boldsymbol{f}_{j,\tau}\log\left(\frac{\boldsymbol{f}_{j,\tau}}{\boldsymbol{v}_{j,\tau}}\right)\right),\end{aligned}$$

where $\boldsymbol{v}_{j,\tau} = \frac{1}{2}\left(\widehat{\boldsymbol{f}}_{j,\tau} + \boldsymbol{f}_{j,\tau}\right)$.

The experimental results, shown in Figure 9, are consistent with that observed under the *AMSE* metric in Figure 5.



(a) Foursquare (b) Sin (c) Log (d) Foursquare (e) Sin (f) Log

Fig. 8. The running time with ϵ or w varied.

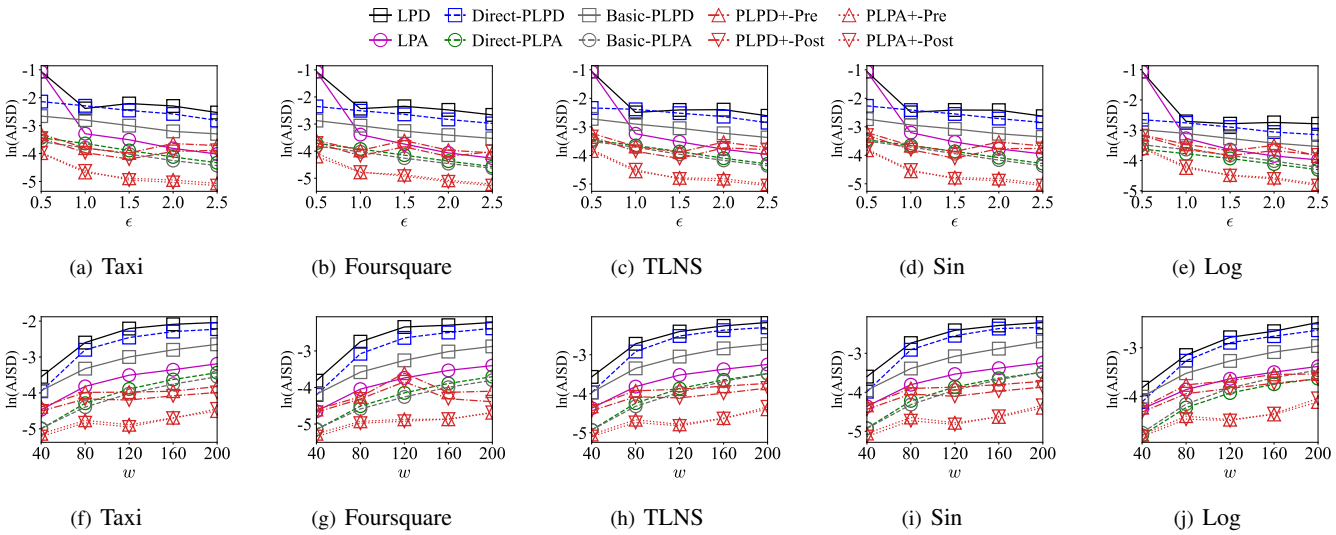
3) Ablation Study: Figure 10 reports the *AMSE* ablation on Foursquare, Sin, and Log, and Figure 11 reports the AJSD ablation on all datasets; the trends are consistent with Figures 6 and 10.

H. Algorithm Details

1) Full PLPD/PLPA*:* Algorithm 8 gives the full version of Algorithm 2, including the PFO.P/A/E steps.

2) Full PAA-CW: Algorithm 9 details how PAA-CW in Algorithm 4 instantiates the auxiliary inputs of Table III for active and reused groups.

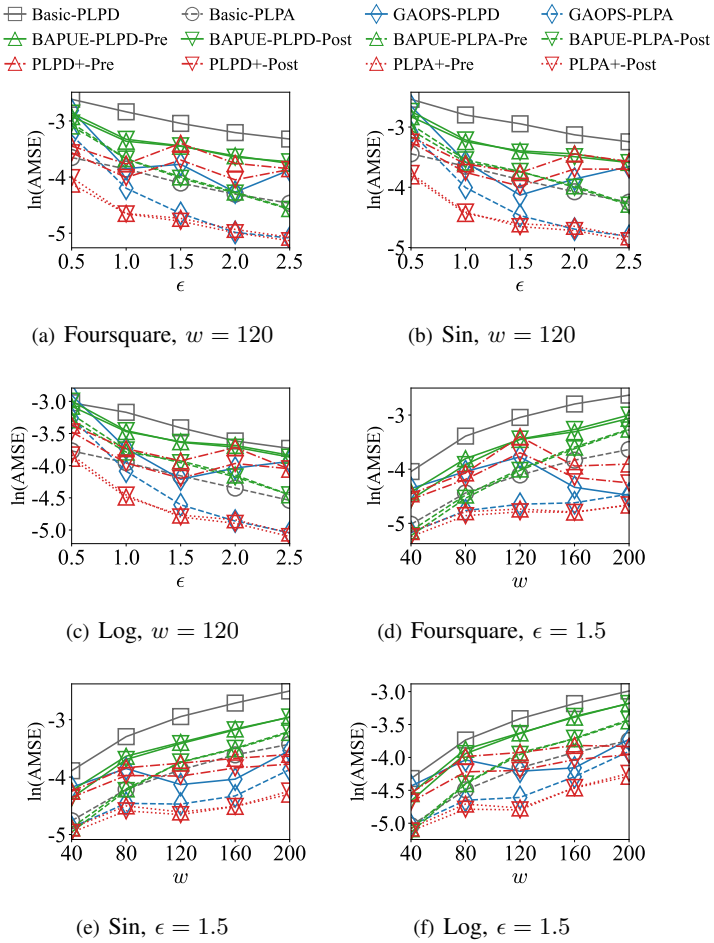
3) Ablation Algorithms for PLPD$^+$ *and* PLPA$^+$*:* Algorithms 10–13 give the GA-OPS-only and BA-PUE-only variants used in Section VI-E.



(a) Taxi (b) Foursquare (c) TLNS (d) Sin (e) Log

(f) Taxi (g) Foursquare (h) TLNS (i) Sin (j) Log

Fig. 9. $AJSD$ with ϵ or w varied.

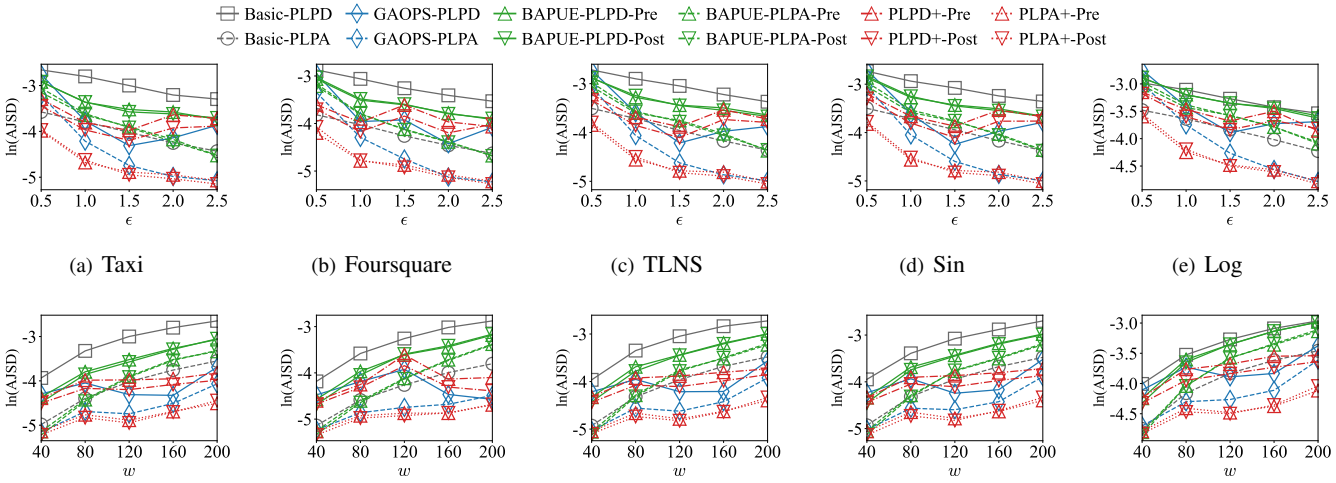


(a) Foursquare, $w = 120$ (b) Sin, $w = 120$

(c) Log, $w = 120$ (d) Foursquare, $\epsilon = 1.5$

(e) Sin, $\epsilon = 1.5$ (f) Log, $\epsilon = 1.5$

Fig. 10. The ablation study of $AMSE$ with different ϵ or w.



(a) Taxi (b) Foursquare (c) TLNS (d) Sin (e) Log

(f) Taxi (g) Foursquare (h) TLNS (i) Sin (j) Log

Fig. 11. The ablation study of $AJSD$ with ϵ or w varied.

Algorithm 8: Unified PLPD/PLPA Stream Estimation

Input: Available users U_A; privacy requirements ($\boldsymbol{w}$, $\boldsymbol{\epsilon}$); data publication history $(\widehat{\boldsymbol{f}}_1, \widehat{\boldsymbol{f}}_2, \ldots, \widehat{\boldsymbol{f}}_{t-1})$; mechanism mode $\mathsf{mode} \in \{\mathrm{PLPD}, \mathrm{PLPA}\}$

Output: The current estimated frequency $\widehat{\boldsymbol{f}}_t$

Calculate distinct privacy budgets $\widetilde{\boldsymbol{\epsilon}}$ and group statistics $\boldsymbol{g}$;
// sub-mechanism $\mathcal{M}_{dis,t}$
Calculate the sampling size $\boldsymbol{z} \leftarrow \left(\lfloor \frac{n}{2w_1} \rfloor, \lfloor \frac{n}{2w_2} \rfloor, \ldots, \lfloor \frac{n}{2w_n} \rfloor\right)$;
Calculate $(z_{opt}, \tau_{opt}, err_{opt}, \boldsymbol{\epsilon}_{opt}) \leftarrow \mathrm{OPS}(\boldsymbol{z}, \boldsymbol{\epsilon}, \boldsymbol{w}, 1/2, d)$ by Algorithm 1;
Sample user set $U_{dis,t}$, whose data is $D_{dis,t}$ and privacy budget list is $\boldsymbol{\epsilon}'_{opt} \subseteq \boldsymbol{\epsilon}_{opt}$, from U_A with the size of z_{opt} and remove $U_{dis,t}$ from U_A, i.e., $U_A = U_A \backslash U_{dis,t}$;
$\widetilde{\boldsymbol{\epsilon}}'_{opt} \leftarrow \mathrm{Unique}(\boldsymbol{\epsilon}'_{opt})$;
Get report $\boldsymbol{Y}_{dis,t} \leftarrow \mathrm{PFO.P}(D_{dis,t})$ with privacy budget list $\boldsymbol{\epsilon}'_{opt}$ from users in $U_{dis,t}$;
Get $\boldsymbol{h}_{dis,t,b} \leftarrow \mathrm{PFO.A}(\boldsymbol{Y}_{dis,t})$ for each group G_b;
Estimate $\widehat{\boldsymbol{f}}_{dis,t} \leftarrow \mathrm{PFO.E}(\boldsymbol{h}_{dis,t}, \widetilde{\boldsymbol{\epsilon}}'_{opt})$;
Calculate $dis \leftarrow \frac{1}{d}\sum_{j=1}^{d}\left(\widehat{\boldsymbol{f}}_{dis,t}[j] - \widehat{\boldsymbol{f}}_l[j]\right)^2 - \frac{1}{d}\sum_{j=1}^{d}\mathrm{Var}\left[\widehat{\boldsymbol{f}}_{dis,t}[j]\right]$;
// sub-mechanism $\mathcal{M}_{pub,t}$
if $\mathsf{mode} = \mathrm{PLPD}$ **then**
  Calculate the optimal remaining population size $n_{r,opt} \leftarrow n/2 - \sum_{\tau=t-\tau_{opt}+1}^{t-1} |U_{pub,\tau}|$;
  Set the optimal number of potential publication users $n_{pp,opt} \leftarrow \lfloor n_{r,opt}/2 \rfloor$;
else
  Calculate the time slots to be nullified $t_N = \frac{\left|U_{pub,l}\right|}{z_{opt}} - 1$;
  if $t - l \le t_N$ **then**
    $\widehat{\boldsymbol{f}}_t \leftarrow \widehat{\boldsymbol{f}}_{t-1}$;
    return $\widehat{\boldsymbol{f}}_{pub,t}$.
  Calculate time slots that can be absorbed $t_A = t - l - t_N$;
  Set the number of potential publication users $n_{pp,opt} \leftarrow z_{opt} \cdot \min(t_A, \tau_{opt})$;
Sample a user set $U_{pub,t}$ from U_A with the size of $|U_{pub,t}| = n_{pp,opt}$ and privacy budget list $\boldsymbol{\epsilon}''_{opt} \subseteq \boldsymbol{\epsilon}_{opt}$;
Calculate the budget group user statistic $\boldsymbol{g}_{pp,opt}$ of $\widetilde{\boldsymbol{\epsilon}}''_{opt} = \mathrm{Unique}(\boldsymbol{\epsilon}''_{opt})$ by $U_{pub,t}$;
Calculate group-level sampling size group $\boldsymbol{z}_{pp}$, where $z_{pp,r} \leftarrow n_{pp,opt} \cdot g_{pp,opt,r}$;
Calculate the potential reporting error $err \leftarrow err\left(\widetilde{\boldsymbol{\epsilon}}''_{opt}, n, \boldsymbol{z}_{pp}, d\right)$ by Equation (1);
if $dis > err$ *and* $n_{pp,opt} \ge u_{min}$ **then**
  Remove $U_{pub,t}$ from U_A, i.e., $U_A \leftarrow U_A \backslash U_{pub,t}$;
  Get report $\boldsymbol{Y}_{pub,t} \leftarrow \mathrm{PFO.P}(D_t)$ with privacy budget list $\boldsymbol{\epsilon}''_{opt}$ from users in $U_{pub,t}$;
  Get $\boldsymbol{h}_{pub,t,r} \leftarrow \mathrm{PFO.A}(\boldsymbol{Y}_{pub,t})$ for each group G_r;
  Calculate $\widehat{\boldsymbol{f}}_{pub,t} \leftarrow \mathrm{PFO.E}(\boldsymbol{h}_{pub,t}, \widetilde{\boldsymbol{\epsilon}}''_{opt})$;
else
  Set $\widehat{\boldsymbol{f}}_{pub,t} \leftarrow \widehat{\boldsymbol{f}}_{pub,t\text{-}1}$;
if $t \ge \tau_{opt}$ **then**
  $U_A \leftarrow U_A \cup U_{dis,t-\tau_{opt}+1} \cup U_{pub,t-\tau_{opt}+1}$;
return $\widehat{\boldsymbol{f}}_{pub,t}$

Algorithm 9: Partial-Absorption-Aware Corrected Weight (PAA-CW)

Input: Mixed publication estimates $\widehat{\boldsymbol{F}}_{pub,t}$; current dissimilarity estimates $\widehat{\boldsymbol{F}}_{dis,t}$; publication-candidate variances $\widehat{\boldsymbol{V}}_{pub,t}$; dissimilarity variances $\widehat{\boldsymbol{V}}_{dis,t}$; reused groups $\boldsymbol{PG}_t^{(R)}$; source sets $\boldsymbol{\mathcal{S}}$; clipping threshold μ; fusion mode $\mathsf{mode}_{fsn} \in \{\mathrm{Pre}, \mathrm{POST}\}$

Output: The fusion-weight matrix $\boldsymbol{\Lambda}^*$

Initialize $\boldsymbol{\Lambda}^* \leftarrow \emptyset$;
for $r \in [K]$ **do**
  if $PG_r \in \boldsymbol{PG}_t^{(R)}$ **then**
    $\tau_r \leftarrow l_r$; $\widehat{\boldsymbol{f}}^{(r)}_{est,\tau_r} \leftarrow \widehat{\boldsymbol{f}}^{(r)}_{pub,l_r}$; $V_{est,r,\tau_r} \leftarrow \widehat{V}_{pub,r,l_r}$;
  else
    $\tau_r \leftarrow t$;
    if $mode_{fsn} = \mathrm{Pre}$ **then**
      $\widehat{\boldsymbol{f}}^{(r)}_{est,\tau_r} \leftarrow \widehat{\boldsymbol{f}}^{(r)}_{dis,t}$; $V_{est,r,\tau_r} \leftarrow \widehat{V}_{dis,r,t}$;
    else
      $\widehat{\boldsymbol{f}}^{(r)}_{est,\tau_r} \leftarrow \widehat{\boldsymbol{f}}^{(r)}_{pub,t}$; $V_{est,r,\tau_r} \leftarrow \widehat{V}_{pub,r,t}$;
for $H \in [K]$ **do**
  Set $\mathcal{C}_H \leftarrow \{H\} \cup \mathcal{S}_H$;
  if $PG_H \in \boldsymbol{PG}_t^{(R)}$ **then**
    for $r \in \mathcal{C}_H$ **do**
      Calculate $\widehat{\boldsymbol{\xi}}_{r|H,t} \leftarrow \widehat{\boldsymbol{f}}^{(r)}_{est,\tau_r} - \widehat{\boldsymbol{f}}^{(H)}_{dis,t}$;
    for $r, s \in \mathcal{C}_H$ **do**
      Set $\widehat{\boldsymbol{M}}_{H,t}(r,s) \leftarrow \left\langle \widehat{\boldsymbol{\xi}}_{r|H,t}, \widehat{\boldsymbol{\xi}}_{s|H,t} \right\rangle - \widehat{V}_{dis,H,t} + \left(\widehat{V}_{pub,r,\tau_r} - V_{est,r,\tau_r}\right)\delta_{r,s}$;
  else
    for $r \in \mathcal{S}_H$ **do**
      Calculate $\widehat{\boldsymbol{\xi}}_{r|H,t} \leftarrow \widehat{\boldsymbol{f}}^{(r)}_{est,\tau_r} - \widehat{\boldsymbol{f}}^{(H)}_{est,t}$;
    for $r, s \in \mathcal{S}_H$ **do**
      Set $\widehat{\boldsymbol{A}}_{H,t}(r,s) \leftarrow \left\langle \widehat{\boldsymbol{\xi}}_{r|H,t}, \widehat{\boldsymbol{\xi}}_{s|H,t} \right\rangle - V_{est,H,t} + \left(\widehat{V}_{pub,r,\tau_r} - V_{est,r,\tau_r}\right)\delta_{r,s}$;
    $\widehat{\boldsymbol{M}}_{H,t} \leftarrow \begin{pmatrix} \widehat{V}_{pub,H,t} & \boldsymbol{0}^\top \\ \boldsymbol{0} & \widehat{\boldsymbol{A}}_{H,t} \end{pmatrix}$;
  $\widetilde{\boldsymbol{M}}_{H,t} \leftarrow \mathrm{PDEC}(\widehat{\boldsymbol{M}}_{H,t}, \mu)$;
  Obtain $\boldsymbol{\lambda}^*_H$ by solving the simplex-constrained quadratic program defined in Section V-B ;
  Add $\boldsymbol{\lambda}^*_H$ to $\boldsymbol{\Lambda}^*$;
return $\boldsymbol{\Lambda}^*$

Algorithm 10: GAOPS-PLPD

Input: Available user set U_A; privacy requirement $(\boldsymbol{w}, \boldsymbol{\epsilon})$ of all Users U; data domain size d; historical data publication $\left(\widehat{\boldsymbol{f}}_1, \widehat{\boldsymbol{f}}_2, \ldots, \widehat{\boldsymbol{f}}_{t-1}\right)$

Output: $\widehat{\boldsymbol{f}}_t$

$\widehat{\boldsymbol{\mathcal{V}}}_{dis,t} \leftarrow \emptyset$; $\widehat{\boldsymbol{F}}_{dis,t} \leftarrow \emptyset$; $\boldsymbol{\mathcal{V}}^{\mathrm{Pre}}_{pub,t} \leftarrow \emptyset$; $\widehat{\boldsymbol{\mathcal{V}}}_{pub,t} \leftarrow \emptyset$; $\widehat{\boldsymbol{F}}_{pub,t} \leftarrow \emptyset$;

Get the privacy group set $\boldsymbol{PG} = \{PG_r = (\widetilde{w}_r, \widetilde{\epsilon}_r, U_r) \mid r \in [K]\}$ with privacy group triple set $\boldsymbol{PGT} = \{PGT_r = (\widetilde{w}_r, \widetilde{\epsilon}_r, n_r) \mid r \in [K]\}$;

// sub-mechanism $\mathcal{M}_{dis,t}$

Get optimal group-aware parameters $(\boldsymbol{z}^*_{opt,dis}, \boldsymbol{\tau}^*, \widetilde{\boldsymbol{\epsilon}}') \leftarrow \text{GA-OPS}(\boldsymbol{PGT}, \frac{1}{2}, d)$ by Algorithm 3;

for $PG_r \in \boldsymbol{PG}$ **do**

  Sample user set $U_{dis,r,t}$, whose data is $D_{dis,r,t}$ and effective privacy budget is $\widetilde{\epsilon}'_r$, from $U_{A,r}$ with the size of $z^*_{opt,dis,r}$ and remove $U_{dis,r,t}$ from $U_{A,r}$, i.e., $U_{A,r} = U_{A,r} \backslash U_{dis,r,t}$;

  Get frequency estimation $\widehat{\boldsymbol{f}}^{(r)}_{dis,t} \leftarrow \text{FO}(U_{dis,r,t}, \widetilde{\epsilon}'_r)$ from users in $U_{dis,r,t}$ and calculate the unbiased vector-level total variance estimate $\widehat{V}_{dis,r,t}$ by Theorem V.1;

  Add $\widehat{\boldsymbol{f}}^{(r)}_{dis,t}$ to $\widehat{\boldsymbol{F}}_{dis,t}$ and $\widehat{V}_{dis,r,t}$ to $\widehat{\boldsymbol{\mathcal{V}}}_{dis,t}$;

Get the population-weighted global estimation $\widehat{\boldsymbol{f}}_{dis,t} \leftarrow \sum_{r\in[K]} \pi_r \widehat{\boldsymbol{f}}^{(r)}_{dis,t}$;

Calculate $dis \leftarrow \frac{1}{d}\sum_{j=1}^{d}\left(\widehat{\boldsymbol{f}}_{dis,t}[j] - \widehat{\boldsymbol{f}}_t[j]\right)^2 - \frac{1}{d}\sum_{r=1}^{K}\pi_r^2 \widehat{V}_{dis,r,t}$;

// sub-mechanism $\mathcal{M}_{pub,t}$

for $PG_r \in \boldsymbol{PG}$ **do**

  Calculate the optimal remaining population size $n_{pub,r} \leftarrow \lfloor n_r/2 \rfloor - \sum_{\tau=t-\tau^*_r+1}^{t-1} |U_{pub,r,\tau}|$;

  Set the optimal number of potential publication users $n_{pp,r} \leftarrow \left\lfloor n_{pub,r}/2 \right\rfloor$;

  $V^{\mathrm{Pre}}_{pub,r,t} \leftarrow V_{smpl}\left(\left\lfloor \frac{n_r}{2} \right\rfloor, n_{pp,r}\right) + V_{LDP}\left(\epsilon'_r, n_{pp,r}, d\right)$;

  Add $V^{\mathrm{Pre}}_{pub,r,t}$ to $\boldsymbol{\mathcal{V}}^{\mathrm{Pre}}_{pub,t}$;

$\widetilde{err}^{+} \leftarrow \frac{1}{d}\sum_{r=1}^{K}\pi_r^2 V^{\mathrm{Pre}}_{pub,r,t}$;

if $dis > \widetilde{err}^{+}$ *and* $\min_{r\in[K]} n_{pp,r} \geq u_{min}$ **then**

  for $PG_r \in \boldsymbol{PG}$ **do**

    Sample a user set $U_{pub,r,t}$ from $U_{A,r}$ with the size of $|U_{pub,r,t}| = n_{pp,r}$;

    Remove $U_{pub,r,t}$ from $U_{A,r}$, i.e., $U_{A,r} \leftarrow U_{A,r} \backslash U_{pub,r,t}$;

    Get frequency estimation $\widehat{\boldsymbol{f}}^{(r)}_{pub,t} \leftarrow \text{FO}(U_{pub,r,t}, \widetilde{\epsilon}'_r)$ from users in $U_{pub,r,t}$ and calculate its unbiased vector-level variance estimate $\widehat{V}_{pub,r,t}$;

    Add $\widehat{\boldsymbol{f}}^{(r)}_{pub,t}$ to $\widehat{\boldsymbol{F}}_{pub,t}$ and $\widehat{V}_{pub,r,t}$ to $\widehat{\boldsymbol{\mathcal{V}}}_{pub,t}$;

  Get the population-weighted global estimation $\widehat{\boldsymbol{f}}_{pub,t} \leftarrow \sum_{r\in[K]} \pi_r \widehat{\boldsymbol{f}}^{(r)}_{pub,t}$;

else

  Set $\widehat{\boldsymbol{f}}_{pub,t} \leftarrow \widehat{\boldsymbol{f}}_{pub,t\text{-}1}$;

for $PG_r \in \boldsymbol{PG}$ **do**

  if $t \geq \tau^*_r$ **then**

    $U_{A,r} \leftarrow U_{A,r} \cup U_{dis,r,t-\tau^*_r+1} \cup U_{pub,r,t-\tau^*_r+1}$;

return $\widehat{\boldsymbol{f}}_{pub,t}$.

Algorithm 11: GAOPS-PLPA

Input: Available user set U_A; privacy requirement $(\boldsymbol{w}, \boldsymbol{\epsilon})$ of all Users U; data domain size d; historical data publication $\left(\widehat{\boldsymbol{f}}_1, \widehat{\boldsymbol{f}}_2, \ldots, \widehat{\boldsymbol{f}}_{t-1}\right)$

Output: $\widehat{\boldsymbol{f}}_t$

1 The same as Lines 1–2 in Algorithm 10;

// sub-mechanism $\mathcal{M}_{dis,t}$

2 The same as Lines 3-9 in Algorithm 10;

// sub-mechanism $\mathcal{M}_{pub,t}$

3 Initialize active group set $\boldsymbol{PG}_t^{(A)} \leftarrow \emptyset$, reused group set $\boldsymbol{PG}_t^{(R)} \leftarrow \emptyset$;

4 **for** $PG_r \in \boldsymbol{PG}$ **do**

5 Calculate the time slots to be nullified $t_{N,r} = \frac{\left|U_{pub,r,l_r}\right|}{z^*_{opt,dis,r}} - 1$;

6 **if** $t - l_r \leq t_{N,r}$ **then**

7 Add $\widehat{\boldsymbol{f}}^{(r)}_{pub,l_r}$ to $\widehat{\boldsymbol{F}}_{pub,t}$, $\widehat{V}_{pub,r,l_r}$ to $\boldsymbol{\mathcal{V}}^{\mathrm{Pre}}_{pub,t}$ and $\widehat{\boldsymbol{\mathcal{V}}}_{pub,t}$, and $\boldsymbol{PG}_r$ to $\boldsymbol{PG}_t^{(R)}$;

8 **else**

9 Calculate group-aware time slots that can be absorbed $t_{A,r} = t - l_r - t_{N,r}$;

10 Set the number of potential publication users $n_{pp,r} \leftarrow z^*_{opt,dis,r} \cdot \min(t_{A,r}, \tau^*_r)$;

11 $V^{\mathrm{Pre}}_{pub,r,t} \leftarrow \left(V_{smpl}\left(\left\lfloor \frac{n_r}{2} \right\rfloor, n_{pp,r}\right) + V_{LDP}\left(\epsilon'_r, n_{pp,r}, d\right)\right)$;

12 Add $V^{\mathrm{Pre}}_{pub,r,t}$ to $\boldsymbol{\mathcal{V}}^{\mathrm{Pre}}_{pub,t}$;

13 Add PG_r to $\boldsymbol{PG}_t^{(A)}$;

14 **if** $\boldsymbol{PG}_t^{(A)} = \emptyset$ **then**

15 Set $\widehat{\boldsymbol{f}}_{pub,t} \leftarrow \widehat{\boldsymbol{f}}_{pub,t-1}$;

16 **else**

17 Calculate the staleness-aware potential reporting error $\widetilde{err}^+$ by Equation (9) with $\gamma_r = \pi_r, \forall r$;

18 **if** $dis > \widetilde{err}^+$ *and* $\min_{PG_r \in \boldsymbol{PG}_t^{(A)}} n_{pp,r} \geq u_{min}$ **then**

19 **for** $PG_r \in \boldsymbol{PG}_t^{(A)}$ **do**

20 Sample a user set $U_{pub,r,t}$ from $U_{A,r}$ with the size of $|U_{pub,r,t}| = n_{pp,r}$;

21 Remove $U_{pub,r,t}$ from $U_{A,r}$, i.e., $U_{A,r} \leftarrow U_{A,r} \backslash U_{pub,r,t}$;

22 Get frequency estimation $\widehat{\boldsymbol{f}}^{(r)}_{pub,t} \leftarrow \mathrm{FO}(U_{pub,r,t}, \widetilde{\epsilon}'_r)$ from users in $U_{pub,r,t}$ and calculate its unbiased vector-level variance $\widehat{V}_{pub,r,t}$;

23 Add $\widehat{\boldsymbol{f}}^{(r)}_{pub,t}$ to $\widehat{\boldsymbol{F}}_{pub,t}$, and $\widehat{V}_{pub,r,t}$ to $\widehat{\boldsymbol{\mathcal{V}}}_{pub,t}$;

24 Get the population-weighted global estimation $\widehat{\boldsymbol{f}}_{pub,t} \leftarrow \sum_{r \in [K]} \pi_r \widehat{\boldsymbol{f}}^{(r)}_{pub,t}$;

25 **else**

26 Set $\widehat{\boldsymbol{f}}_{pub,t} \leftarrow \widehat{\boldsymbol{f}}_{pub,t\text{-}1}$;

27 **for** $PG_r \in \boldsymbol{PG}$ **do**

28 **if** $t \geq \tau^*_r$ **then**

29 $U_{A,r} \leftarrow U_{A,r} \cup U_{dis,r,t-\tau^*_r+1} \cup U_{pub,r,t-\tau^*_r+1}$;

30 **return** $\widehat{\boldsymbol{f}}_{pub,t}$.

Algorithm 12: BAPUE-PLPD

Input: Available user set U_A; privacy requirement $(\boldsymbol{w}, \boldsymbol{\epsilon})$ of all Users U; data domain size d; the eigenvalue lowbound μ; historical data publication $\left(\widehat{\boldsymbol{f}}_1, \widehat{\boldsymbol{f}}_2, \ldots, \widehat{\boldsymbol{f}}_{t-1}\right)$; fusion mode $\mathsf{mode}_{fsn} \in \{\mathrm{Pre}, \mathrm{POST}\}$

Output: $\widehat{\boldsymbol{f}}_t$

$\widehat{\boldsymbol{\mathcal{V}}}_{dis,t} \leftarrow \emptyset$; $\widehat{\boldsymbol{F}}_{dis,t} \leftarrow \emptyset$; $\boldsymbol{\mathcal{V}}^{\mathrm{Pre}}_{pub,t} \leftarrow \emptyset$; $\widehat{\boldsymbol{\mathcal{V}}}_{pub,t} \leftarrow \emptyset$; $\boldsymbol{\Lambda}^{\mathrm{Pre}}_{pub,t} \leftarrow \emptyset$; $\widehat{\boldsymbol{F}}_{pub,t} \leftarrow \emptyset$;

Get the privacy group set $\boldsymbol{PG} = \{PG_r = (\widetilde{w}_r, \widetilde{\epsilon}_r, U_r) \mid r \in [K]\}$ with privacy group triple set $\boldsymbol{PGT} = \{PGT_r = (\widetilde{w}_r, \widetilde{\epsilon}_r, n_r) \mid r \in [K]\}$;

Construct the candidate source sets $\{\mathcal{S}_r \mid r \in [K]\}$ for all groups;

// sub-mechanism $\mathcal{M}_{dis,t}$

$\boldsymbol{z} \leftarrow \left(\left\lfloor \frac{n}{2w_1} \right\rfloor, \left\lfloor \frac{n}{2w_2} \right\rfloor, \ldots, \left\lfloor \frac{n}{2w_n} \right\rfloor\right)$;

Get the optimal population size, recycle window size, error and budget $z_{opt}, \tau_{opt}, err_{opt}, \boldsymbol{\epsilon}'_{opt} \leftarrow \mathrm{OPS}(\boldsymbol{z}, \boldsymbol{\epsilon}, \boldsymbol{w}, 1/2, d)$ by Algorithm 1;

Get optimal group privacy budget $\widetilde{\boldsymbol{\epsilon}}'$ by $\boldsymbol{PG}$ and $\boldsymbol{\epsilon}'$;

Set $\left(z^*_{opt,dis,1} \cdots z^*_{opt,dis,K}\right) \leftarrow \mathrm{LargestRemainder}\left(z_{opt}, \boldsymbol{\pi}\right)$;

for $PG_r \in \boldsymbol{PG}$ **do**

  Sample user set $U_{dis,r,t}$, whose data is $D_{dis,r,t}$, from $U_{A,r}$ with the size of $z^*_{opt,dis,r}$ and remove $U_{dis,r,t}$ from $U_{A,r}$, i.e., $U_{A,r} = U_{A,r} \backslash U_{dis,r,t}$;

  Get frequency estimation $\widehat{\boldsymbol{f}}^{(r)}_{dis,t} \leftarrow \mathrm{FO}(U_{dis,r,t}, \widetilde{\epsilon}'_r)$ from users in $U_{dis,r,t}$ and calculate the unbiased vector-level total variance estimate $\widehat{V}_{dis,r,t}$ by Theorem V.1;

  Add $\widehat{\boldsymbol{f}}^{(r)}_{dis,t}$ to $\widehat{\boldsymbol{F}}_{dis,t}$ and $\widehat{V}_{dis,r,t}$ to $\widehat{\boldsymbol{\mathcal{V}}}_{dis,t}$;

Get the population-weighted global estimation $\widehat{\boldsymbol{f}}_{dis,t} \leftarrow \sum_{r\in[K]} \pi_r \widehat{\boldsymbol{f}}^{(r)}_{dis,t}$;

Calculate $dis \leftarrow \frac{1}{d}\sum_{j=1}^{d}\left(\widehat{\boldsymbol{f}}_{dis,t}[j] - \widehat{\boldsymbol{f}}_t[j]\right)^2 - \frac{1}{d}\sum_{r=1}^{K}\pi_r^2 \widehat{V}_{dis,r,t}$;

// sub-mechanism $\mathcal{M}_{pub,t}$

for $PG_r \in \boldsymbol{PG}$ **do**

  Calculate the optimal remaining population size $n_{pub,r} \leftarrow \lfloor n_r/2 \rfloor - \sum_{\tau=t-\tau_{opt}+1}^{t-1} |U_{pub,r,\tau}|$;

  Set the optimal number of potential publication users $n_{pp,r} \leftarrow \left\lfloor n_{pub,r}/2 \right\rfloor$;

  $V^{\mathrm{Pre}}_{pub,r,t} \leftarrow V_{smpl}\left(\left\lfloor \frac{n_r}{2} \right\rfloor, n_{pp,r}\right) + V_{LDP}\left(\epsilon'_r, n_{pp,r}, d\right)$;

  Add $V^{\mathrm{Pre}}_{pub,r,t}$ to $\boldsymbol{\mathcal{V}}^{\mathrm{Pre}}_{pub,t}$;

$\boldsymbol{\Lambda}^{\mathrm{Pre}}_{pub,t} \leftarrow \text{PAA-CW}\left(\widehat{\boldsymbol{F}}_{pub,t}, \widehat{\boldsymbol{F}}_{dis,t}, \boldsymbol{\mathcal{V}}^{\mathrm{Pre}}_{pub,t}, \boldsymbol{\mathcal{V}}_{dis,t}, \emptyset, \boldsymbol{\mathcal{S}}, \mu, \mathrm{Pre}\right)$ by Algorithm 9;

Calculate the potential publication error $\widetilde{err}^{+}$ according to Equation (10);

if $dis > \widetilde{err}^{+}$ *and* $\min_{r\in[K]} n_{pp,r} \geq u_{min}$ **then**

  for $PG_r \in \boldsymbol{PG}$ **do**

    Sample a user set $U_{pub,r,t}$ from $U_{A,r}$ with the size of $|U_{pub,r,t}| = n_{pp,r}$;

    Remove $U_{pub,r,t}$ from $U_{A,r}$, i.e., $U_{A,r} \leftarrow U_{A,r} \backslash U_{pub,r,t}$;

    Get frequency estimation $\widehat{\boldsymbol{f}}^{(r)}_{pub,t} \leftarrow \mathrm{FO}(U_{pub,r,t}, \widetilde{\epsilon}'_r)$ from users in $U_{pub,r,t}$ and calculate its unbiased vector-level variance estimate $\widehat{V}_{pub,r,t}$;

    Add $\widehat{\boldsymbol{f}}^{(r)}_{pub,t}$ to $\widehat{\boldsymbol{F}}_{pub,t}$ and $\widehat{V}_{pub,r,t}$ to $\widehat{\boldsymbol{\mathcal{V}}}_{pub,t}$;

  if $\mathsf{mode}_{fsn} = \mathrm{POST}$ **then**

    $\boldsymbol{\Lambda}^{(post)}_{pub,t} \leftarrow \text{PAA-CW}\left(\widehat{\boldsymbol{F}}_{pub,t}, \widehat{\boldsymbol{F}}_{dis,t}, \widehat{\boldsymbol{\mathcal{V}}}_{pub,t}, \widehat{\boldsymbol{\mathcal{V}}}_{dis,t}, \emptyset, \boldsymbol{\mathcal{S}}, \mu, \mathrm{POST}\right)$ by Algorithm 9;

    Set $\boldsymbol{\Lambda}^{(fnl)}_{pub,t} \leftarrow \boldsymbol{\Lambda}^{(post)}_{pub,t}$;

  else

    Set $\boldsymbol{\Lambda}^{(fnl)}_{pub,t} \leftarrow \boldsymbol{\Lambda}^{\mathrm{Pre}}_{pub,t}$;

  Fuse $\widetilde{\boldsymbol{F}}_{pub,t}$ by Equation (2) with $\boldsymbol{\Lambda}^{(fnl)}_{t}$;

  Get the population-weighted global estimation $\widehat{\boldsymbol{f}}_{pub,t} \leftarrow \sum_{r\in[K]} \pi_r \widetilde{\boldsymbol{f}}^{(r)}_{pub,t}$;

else

  Set $\widehat{\boldsymbol{f}}_{pub,t} \leftarrow \widehat{\boldsymbol{f}}_{pub,t\text{-}1}$;

if $t \geq \tau_{opt}$ **then**

  for $PG_r \in \boldsymbol{PG}$ **do**

    $U_{A,r} \leftarrow U_{A,r} \cup U_{dis,r,t-\tau_{opt}+1} \cup U_{pub,r,t-\tau_{opt}+1}$;

return $\widehat{\boldsymbol{f}}_{pub,t}$;

Algorithm 13: BAPUE-PLPA

Input: Available user set U_A; privacy requirement $(\boldsymbol{w}, \boldsymbol{\epsilon})$ of all Users U; data domain size d; the eigenvalue lowbound μ; historical data publication $\left(\widehat{\boldsymbol{f}}_1, \widehat{\boldsymbol{f}}_2, \ldots, \widehat{\boldsymbol{f}}_{t-1}\right)$; fusion mode $\mathsf{mode}_{fsn} \in \{\mathrm{Pre}, \mathrm{POST}\}$

Output: $\widehat{\boldsymbol{f}}_t$

The same as Lines 1–3 in Algorithm 12;

// sub-mechanism $\mathcal{M}_{dis,t}$

The same as Lines 5-13 in Algorithm 12;

// sub-mechanism $\mathcal{M}_{pub,t}$

Initialize active group set $\boldsymbol{PG}_t^{(A)} \leftarrow \emptyset$, reused group set $\boldsymbol{PG}_t^{(R)} \leftarrow \emptyset$;

for $PG_r \in \boldsymbol{PG}$ **do**

  Calculate the time slots to be nullified $t_{N,r} = \frac{\left|U_{pub,r,l_r}\right|}{z^*_{opt,dis,r}} - 1$;

  if $t - l_r \leq t_{N,r}$ **then**

    Add $\widehat{\boldsymbol{f}}^{(r)}_{pub,l_r}$ to $\widehat{\boldsymbol{F}}_{pub,t}$, $\widehat{V}_{pub,r,l_r}$ to $\boldsymbol{\mathcal{V}}^{\mathrm{Pre}}_{pub,t}$ and $\widehat{\boldsymbol{\mathcal{V}}}_{pub,t}$, and $\boldsymbol{PG}_r$ to $\boldsymbol{PG}_t^{(R)}$;

  else

    Calculate group-aware time slots that can be absorbed $t_{A,r} = t - l_r - t_{N,r}$;

    Set the number of potential publication users $n_{pp,r} \leftarrow z^*_{opt,dis,r} \cdot \min(t_{A,r}, \tau_{opt})$;

    $V^{\mathrm{Pre}}_{pub,r,t} \leftarrow \left(V_{smpl}\left(\left\lfloor \frac{n_r}{2} \right\rfloor, n_{pp,r}\right) + V_{LDP}\left(\epsilon'_r, n_{pp,r}, d\right)\right)$;

    Add $V^{\mathrm{Pre}}_{pub,r,t}$ to $\boldsymbol{\mathcal{V}}^{\mathrm{Pre}}_{pub,t}$;

    Add PG_r to $\boldsymbol{PG}_t^{(A)}$;

if $\boldsymbol{PG}_t^{(A)} = \emptyset$ **then**

  Set $\widehat{\boldsymbol{f}}_{pub,t} \leftarrow \widehat{\boldsymbol{f}}_{pub,t-1}$;

else

  $\boldsymbol{\Lambda}^{\mathrm{Pre}}_{pub,t} \leftarrow \text{PAA-CW}\left(\widehat{\boldsymbol{F}}_{pub,t}, \widehat{\boldsymbol{F}}_{dis,t}, \boldsymbol{\mathcal{V}}^{\mathrm{Pre}}_{pub,t}, \boldsymbol{\mathcal{V}}_{dis,t}, \boldsymbol{PG}_t^{(R)}, \boldsymbol{\mathcal{S}}, \mu, \mathrm{Pre}\right)$ by Algorithm 9;

  Calculate the staleness-aware potential reporting error $\widetilde{err}^+$ by Equation (9);

  if $dis > \widetilde{err}^+$ *and* $\min_{PG_r \in \boldsymbol{PG}_t^{(A)}} n_{pp,r} \geq u_{min}$ **then**

    for $PG_r \in \boldsymbol{PG}_t^{(A)}$ **do**

      Sample a user set $U_{pub,r,t}$ from $U_{A,r}$ with the size of $|U_{pub,r,t}| = n_{pp,r}$;

      Remove $U_{pub,r,t}$ from $U_{A,r}$, i.e., $U_{A,r} \leftarrow U_{A,r} \backslash U_{pub,r,t}$;

      Get frequency estimation $\widehat{\boldsymbol{f}}^{(r)}_{pub,t} \leftarrow \mathrm{FO}(U_{pub,r,t}, \widetilde{\epsilon}'_r)$ from users in $U_{pub,r,t}$ and calculate its unbiased vector-level variance $\widehat{V}_{pub,r,t}$;

      Add $\widehat{\boldsymbol{f}}^{(r)}_{pub,t}$ to $\widehat{\boldsymbol{F}}_{pub,t}$, and $\widehat{V}_{pub,r,t}$ to $\widehat{\boldsymbol{\mathcal{V}}}_{pub,t}$;

    if $\mathsf{mode}_{fsn} = \mathrm{POST}$ **then**

      $\boldsymbol{\Lambda}^{(post)}_{pub,t} \leftarrow \text{PAA-CW}\left(\widehat{\boldsymbol{F}}_{pub,t}, \widehat{\boldsymbol{F}}_{dis,t}, \widehat{\boldsymbol{\mathcal{V}}}_{pub,t}, \widehat{\boldsymbol{\mathcal{V}}}_{dis,t}, \boldsymbol{PG}_t^{(R)}, \boldsymbol{\mathcal{S}}, \mu, \mathrm{POST}\right)$ by Algorithm 9;

      Set $\boldsymbol{\Lambda}^{(fnl)}_{pub,t} \leftarrow \boldsymbol{\Lambda}^{(post)}_{pub,t}$;

    else

      Set $\boldsymbol{\Lambda}^{(fnl)}_{pub,t} \leftarrow \boldsymbol{\Lambda}^{\mathrm{Pre}}_{pub,t}$;

    Fuse $\widetilde{\boldsymbol{F}}_{pub,t}$ by Equation (2) with $\boldsymbol{\Lambda}^{(fnl)}_t$;

    Get the population-weighted global estimation $\widehat{\boldsymbol{f}}_{pub,t} \leftarrow \sum_{r \in [K]} \pi_r \widetilde{\boldsymbol{f}}^{(r)}_{pub,t}$;

  else

    Set $\widehat{\boldsymbol{f}}_{pub,t} \leftarrow \widehat{\boldsymbol{f}}_{pub,t\text{-}1}$;

if $t \geq \tau_{opt}$ **then**

  for $PG_r \in \boldsymbol{PG}$ **do**

    $U_{A,r} \leftarrow U_{A,r} \cup U_{dis,r,t-\tau_{opt}+1} \cup U_{pub,r,t-\tau_{opt}+1}$;

return $\widehat{\boldsymbol{f}}_{pub,t}$.